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2. To supply an additional medium for the publication of expository mathematics.
3. To promote more scientific methods of teaching mathematics.
4. To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

On Control

The man of today who hopes to be considered abreast of his times can now postpone no longer the formation of some views upon the subject of control in its anthropological aspects. Whether its novel features indicate a novel basis, and whether its characteristics fit it specially for success are questions which should engage his curiosity. If there are facts behind the facade, his is the challenge to find them.

What is control? Is it two different things reacting? Upon the one hand the controller, upon the other the controlled—here the finger on the button and there the button? Is it rather a single thing with several aspects? Here the controller compelling the controlled, there the controlled compelling the controller to compel them. If this looks less than clear, recast the sentence as follows: here the elected constraining the electorate, there the electorate constraining the elected. If the pattern has meaning in the one form, can it be meaningless in the other? The first merely substitutes stronger terms (and less squeamish methods) into the structure of the second. To be sure it alters intercommunication also, in place of ballots putting mental telepathy. Superficially, however, control appears to be a mere variation upon familiar systems.

Maybe, on the other hand, the proof of the pudding is in the eating, and control is what control does. When upon one side of an international boundary foreign policy is controlled, control is a blast of heat which reduces the iron frontier to fiction. When upon some venturesome critic a controlled press focuses its disapproval, control is an acid which converts his backbone to jelly. Whatever the human issue, control is the specific to settle it. Control disposes of problems not by solving them, but by eliminating them. It is an answer to every question except itself, and to itself it is no question. It has been observed that control produces consequences, quick and concrete. He who believes that programs of fundamental significance can be advanced in quick particular steps prefers it as leader to others which are better adjusted to theories of human progress molecular in character. Similarly, he who prefers assuming the responsibility for a single vast mistake, to assuming it for a daily series of small ones prefers control.

The virtues of control in education have been too little emphasized. With what contentment would the teacher face his controlled classes, their mental powers intensified by the knowledge that, if they failed, they would complete their education in a concentration camp! With

what contentment would the student face his controlled teachers, their tongues inspired by the knowledge that, if they flunked him, they would share his camp with him! If spurring, literally, students to feverish activity and unleasing in full the energy of teachers is the cure, as some assume, for the so-called defects of current education, then no scheme promises so much improvement as strict control. The advocate of tricking mathematics out in one way or another to ease it over must face the likelihood that his program, in comparison, would be costly to apply and only modestly fruitful. Wherever there are many who want results and want them quickly, of course control will have its day. On the other hand, he who requires to be allowed to live unhampered by it may have to exalt deliberately ideals whose fulfillment he himself can never see, and discard many comfortable immediate ones such as prosperity just around the corner, or peace in our time.

Groups which invest in control run the risk of placing all their eggs in one basket. If not convinced that contiguity toughens eggshells beyond the danger of ordinary shocks, they can at least anticipate a tremendous splash. The trend of the times is away from the solitary ruffian. Toughness is an attribute of gangs, and other aggregates. As country after country (in search of a more perfect union) adopts the paraphernalia of control, as foreign policy and press go hand in hand, as art and education tag along, as the parade grows, a question grows with it. Is this good thing not likely to become too much of a good thing? When all are marching who will be left to cheer? Will there be none at all upon the side lines?

"What a question", says the scientist. "What a question indeed", say physicists and mathematicians. "The subjects which we study are of course beyond control. Their laws are immutable and their forms determined by the nature of things itself. As well expect a theorem to alter its content under compulsion as expect the tides of the Atlantic to obey a traffic light. In the realms of physical and logical fact, control cannot so much as move a comma. The man of today, whether abreast of his times or not, need form no further views upon that point."

To end the discussion upon the high point of the sentences just above would be in agreement with literary principles. High points, however, are likely to be wobbly ones, and an editorial has no business to escort its readers up a shaky pinnacle and then expire. Even when actual instead of hypothetical mathematicians speak in generalities, they are fairly sure to be pussy-footing. The scientist standing rapt, but not silent, in the presence of the mysteries of the universe has tucked his intelligence safely away in its little crib before facing the microphone.

Down, then, from the stratosphere of science, where theorems do not alter under compulsion. At mean sea level theorems change their contents on slight provocation, no more perhaps than some small substitution well behind the scenes. Replace Euclidean space by non-Euclidean, for instance, and see what happens to the theorems of geometry. Of course two paragraphs above the straw-men scientists—now flattened— may have been speaking in terms beyond the reach of substitutions.

It is, however, true that science and in particular mathematics are predisposed toward projects extending beyond the single lifetime. For twenty centuries the status of the postulate of parallels maintained its interest. Many a man has spent his life upon a physical problem whose solution he did not live to see. Unless impelled by that scholastic rivalry between institutions which is a feature of the upper levels of education today, few research workers concern themselves with immediate ends or quick results. A handy scheme for achieving one or the other does not intimately commend itself to them. Mathematics supplies an example of a study not naturally adapted to profit by control. It does not need to join the parade.

The question whether it could be compelled to is something else. After all science needs spokesmen. To control it, what more would there be to do than silence it? Why even trouble to convert a possible dissenter when he can be annihilated? Science can be readily enough eliminated from the side lines. In addition, are we so very sure of the truth itself? When we say that we are interested in the truth, are we not confessing doubts at least about its completeness? Perhaps its nature too can be controlled. Perhaps a blow torch might after all annul the laws of gravitation, or an acid bath convert the theorem of Pythagoras into a tangle of string.

Louisiana State University. NORMAN E. RUTT.

Binomial Congruences

By Emory P. Starke
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I. *Introduction.* In recent years there has been an increasing tendency to organize the more elementary parts of the Theory of Numbers into a course available to college undergraduates. The value of such work can hardly be questioned: the only real problem is the selection of material.* As a topic to be included in such a course, *binomial congruences* can hardly be surpassed in interest and value, but the instructor is handicapped by the inaccessibility in texts and reference works of a number of its most attractive results. It is with the hope that this interesting field may be opened to a larger number of student mathematicians, that the following self-contained summary of materials is presented. It is assumed that the reader is familiar with the most elementary parts of the Theory of Numbers (as given in the usual texts,† including the Euler totient $\varphi(N)$, the Fermat-Euler theorem (*i. e.*, $a^{\varphi(N)} \equiv 1 \bmod N$, if a is prime to N), and the first propositions on congruences. In the present paper all letters are understood to represent non-negative integers.

If in the binomial congruence, $AX^a \equiv B \pmod{M}$, A, B and M are not relatively prime in pairs, the congruence may be transformed easily into one in which the corresponding constants have this property. Thus (A,M)‡ must divide B, for otherwise there are evidently no solutions. If we divide both members of the congruence by (A,B) we have a new congruence $A'X^a \equiv B' \pmod{M'}$, where $A' = A/(A,B)$, $B' = B/(A,B)$ and $M' = M/(A,M)$; thus $(A',B') = 1$ and $(A',M') = 1$.§ If now $(B', M') \neq 1$, we choose the smallest integer d such that (B', M') is a divisor of d^a. Noting that X must then be divisible by d, we may put dx for X, divide both sides of the congruence by (B', d^a) and thus

*A suggested syllabus for a one semester course appeared in this Magazine for November, 1935, pp. 53-57.

†See G. B. Mathews, *Theory of Numbers* (1892), pp. 4-10, 15, 16; R. D. Carmichael, *Theory of Numbers* (1914), pp. 1-21, 30-49; also L. E. Dickson, *Introduction to the Theory of Numbers* (1929), pp. 1-9. These works will hereafter be referred to as M, C, D, respectively. The more elaborate treatises of Bachman and Landau (in German) should be consulted where possible. An admirable and comprehensive account of the development of many of the topics in this paper, with references to original sources, is available in L. E. Dickson, *History of the Theory of Numbers*, Vol. I, pp. 181-222.

‡(A,M) is a symbol for the highest common factor of A and M.

§Of course, there are (A,M) times as many solutions x, $0 \leqq x < M$, for the original congruence as there are solutions x, $0 \leqq x < M'$, for the transformed one. An analogous remark applies to the next division by (B', d^a).

obtain $ax^\alpha \equiv b \pmod m$ in which $a = A'd^\alpha/(B', d^\alpha)$, $b = B'/(B', d^\alpha)$ and $m = M'/(B', M')$ are integers relatively prime in pairs. If we now multiply both members by $a^{\varphi(m)-1}$ we have $a^{\varphi(m)}x^\alpha \equiv a^{\varphi(m)-1}b \pmod m$, which by Fermat's theorem reduces to $x^\alpha \equiv c \pmod m$ where m and $c \equiv a^{\varphi(m)-1}b$ are relatively prime. Hence also x is prime to m. Conversely, from the solutions of this last congruence all solutions of the original congruence are readily obtained.

If now x_0 is a solution of $x^\alpha \equiv c \pmod m$, all the solutions are given by $x_0 y_i$, where y_i are the solutions of $y^\alpha \equiv 1 \pmod m$. Evidently $x_0 y_i$ is a solution. Conversely, if x_1 and x_0 are two solutions: since $(x_0, m) = 1$, we can determine k such that $x_1 \equiv kx_0 \pmod m$; then $x_1^\alpha \equiv k^\alpha x_0^\alpha$. becomes $c \equiv k^\alpha c$ or $1 \equiv k^\alpha$; whence $k \equiv y_i$. Our problem is then resolved into two parts: (i) find all solutions of $y^\alpha \equiv 1 \pmod m$ and (ii) find one solution of $x^\alpha \equiv c \pmod m$. Final results for these two parts are given in sections IV and V, respectively.

II. *Preliminary theorems; exponents, primitive roots.*

(1). *There exist values of x which satisfy simultaneously the linear congruences $x \equiv a_1 \pmod{m_1}$ and $x \equiv a_2 \pmod{m_2}$, where m_1 and m_2 are relatively prime; all these values of x are congruent mod $m_1 m_2$.* The extension to more than two congruences is immediate.*

Proof: Upon substituting $x = km_1 + a_1$ into the second congruence, we have $km_1 + a_1 \equiv a_2 \pmod{m_2}$ which yields a unique solution for k; or if k is put in the form $k_0 + lm_2$, there results $x = m_1 k_0 + lm_1 m_2 + a_1$; hence $x \equiv m_1 k_0 + a_1 \pmod{m_1 m_2}$.

(2). If $a^\alpha \equiv 1 \pmod m$ but $a^\beta \not\equiv 1$ for all $\beta < \alpha$, α is called the *exponent* of $a \pmod m$, and we say *a belongs to the exponent $\alpha \pmod m$*. We may write $\alpha = \exp a \pmod m$.

(3). In particular, if $\varphi(m) = \exp g \pmod m$, g is said to be a *primitive root* $\pmod m$.

(4). *If $\alpha = exp\ a \pmod m$ and $a^\tau \equiv 1 \pmod m$, then τ is a multiple of α*†.

Proof: Put $\tau = q\alpha + \rho$, $0 \leqq \rho < \alpha$. Then $a^\tau \equiv 1 \pmod m$ and $a^\tau \equiv (a^\alpha)^q a^\rho \equiv 1 \cdot a^\rho \equiv a^\rho$ imply $a^\rho \equiv 1 \pmod m$, contrary to (2) unless $\rho = 0$.

(5). *If x is any number prime to m, there is one and only one value of γ, $\gamma \leqq \varphi(m)$, for which $g^\gamma \equiv x \pmod m$.*‡ γ is often called the *index* of x.

*Sometimes called the *Chinese Remainder Theorem*. See M, p. 12; D, pp. 11-12.

†See M, p. 17; D, p. 17; C, p. 62.

‡g is a primitive root as in (3). For those values of m which do not possess primitive roots, the theorem has of course no meaning. We need this theorem only when m is a power of an odd prime; by (24) below, there are always primitive roots for such moduli. See also M, pp. 21-22.

Proof: The $\varphi(m)$ numbers $g, g^2, \ldots, g^{\varphi(m)} \equiv 1$ are prime to m and incongruent each to each, for otherwise $g^u \equiv g^v$ with $v < u \leqq \varphi(m)$ implies $g^{u-v} \equiv 1$ contrary to (4). On the other hand, there are altogether only $\varphi(m)$ incongruent numbers x prime to m. Hence we infer a one-to-one correspondence between the values of x and the g^γ.

(6). *If* $\alpha = exp\ a\ (mod\ m)$, *then* $\alpha/(\alpha,s)$ *is* $exp\ a^s\ (\text{mod } m)$.*

Proof: Let $\beta = \exp a^s$; then $(a^s)^\beta \equiv 1 \pmod m$; thus by (4) β is the smallest number for which $s\beta$ is a multiple of α.

(7). *If* $\alpha_1 = exp\ a\ (mod\ m_1)$ *and* $\alpha_2 = exp\ a\ (mod\ m_2)$, *then* $A = exp\ a\ (mod\ M)$, *where* A *is the lowest common multiple of* α_1 *and* α_2, *and* M *is the lowest common multiple of* m_1 *and* m_2.†

Proof: From $a^{\alpha_1} \equiv 1 \pmod{m_1}$ follows $a^A \equiv 1 \pmod{m_1}$, and thus m_1 is a divisor of $a^A - 1$; in symbols, $m_1 \mid (a^A - 1)$. Likewise $m_2 \mid (a^A - 1)$. Hence $M \mid (a^A - 1)$ or $a^A \equiv 1 \pmod M$. Further, if $a^\beta \equiv 1 \pmod M$, then $a^\beta \equiv 1 \pmod{m_1}$ so that $\alpha_1 \mid \beta$ by (4). Likewise $\alpha_2 \mid \beta$ and hence $A \mid \beta$ so that β is not less than A.

(8). *If* $\alpha_1 = exp\ a_1\ (mod\ m)$ *and* $\alpha_2 = exp\ a_2\ (mod\ m)$ *and if* $(\alpha_1, \alpha_2) = 1$, then $\alpha_1\alpha_2 = exp\ a_1a_2\ (mod\ m)$.‡

Proof: Let $\beta = \exp a_1a_2$. Then $(a_1a_2)^{\alpha_1\alpha_2} = (a_1^{\alpha_1})^{\alpha_2} \cdot (a_2^{\alpha_2})^{\alpha_1} \equiv 1$, so that $\beta \mid \alpha_1\alpha_2$. From $1^{\alpha_1} \equiv [(a_1a_2)^\beta]^{\alpha_1} = (a_1^{\alpha_1})^\beta \cdot a_2^{\alpha_1\beta}$ we have $a_2^{\alpha_1\beta} \equiv 1$, so that $\alpha_2 \mid \alpha_1\beta$. Similarly $\alpha_1 \mid \alpha_2\beta$. Hence $\beta = \alpha_1\alpha_2$.

(9). Gauss has suggested the following procedure§ for obtaining primitive roots to a prime modulus, p. Select any a_1 and find $\alpha_1 = \exp a_1 \pmod p$. If $\alpha_1 \neq p-1$, form the residues of $a_1, a_1^2, \ldots, a_1^{\alpha_1 - 1}$, $a_1^{\alpha_1} \equiv 1$. Select a_2 incongruent to every a_1^j. Exp a_2 is not a divisor of α_1 by (6). Let α_2 be the lowest common multiple of α_1 and exp a_2. If α_2 is not exp a_2, factor $\alpha_2 = f \cdot g$ such that $(f,g) = 1$ and $f \mid \alpha_1$ and $g \mid \exp a_2$. By use of (6) find integers whose exponents are f and g respectively, and then by (8) obtain an integer whose exponent is α_2. We shall then have in either case an integer whose exponent α_2 is greater than α_1. If now $\alpha_2 \neq p-1$, we repeat the process. We thus determine a series of integers each belonging to a higher exponent than the preceding, until ultimately one belongs to the exponent $p-1$. From one primitive root, others are determined by application of (6).

*See D, p. 18; M, pp. 19-21.

†A generalization of theorems of Euler and Erlerus.

‡A theorem of Cauchy; see D, p. 17. If $(a_1, a_2) \neq 1$, then β is a divisor of $a_1a_2/(a_1, a_2)$ and also a multiple of $a_1, a_2/(a_1, a_2)^2$.

§See M, pp. 19-21; or for another method, D, pp. 17-19.

(10). *If $\alpha = exp\ a\ (mod\ p)$ where p is an odd prime, and if μ is the largest integer for which $p^\mu | (a^\alpha - 1)$, then $\alpha p^{s-\mu}$ is exp a (mod p^s) for any integer $s > \mu$.**

Proof: Let $\beta = \exp a \pmod{p^s}$. Then $\alpha | \beta$ by (4). Let $a^\alpha = 1 + hp^\mu$ with $(h, p) = 1$, $\mu \geqq 1$. Let d be prime to p and form $(a^\alpha)^{dp^\gamma} = (1 + hp^\mu)^{dp^\gamma} = 1 + hdp^{\gamma+\mu} +$ terms containing higher powers of p. Then $a^{\alpha dp^\gamma} \not\equiv 1 \pmod{p^{\gamma+\mu+1}}$, but (with $d = 1$) $a^{\alpha p^\gamma} \equiv 1 \pmod{p^{\gamma+\mu}}$. If we take $\gamma = s - \mu - 1$ and $\gamma = s - \mu$ respectively, these results become $a^{\alpha d p^{s-\mu-1}} \not\equiv 1 \pmod{p^s}$ but $a^{\alpha p^{s-\mu}} \equiv 1 \pmod{p^s}$. Thus β is the smallest divisor of $\alpha p^{s-\mu}$ which is a multiple of α but is not a divisor of $\alpha d p^{s-\mu-1}$.

(11). *A primitive root g (mod p) is a primitive root (mod p^s) for any s unless $p^2 | (g^{p-1} - 1)$. in the contrary case $g + p$ is a primitive root (mod p^s).*†

Proof: This is a corollary of (10). Note that in the second case p^2 is not a divisor of $[(g+p)^{p-1} - 1]$; for if we let $g^{p-1} - 1 = kp^2$, then $(g+p)^{p-1} - 1 = (g^{p-1} - 1) + (p-1)g^{p-2}p + (p-1)(p-2)g^{p-3}p^2/2 + \ldots \equiv -g^{p-2}p \pmod{p^2}$.

(12). *If $\alpha | \varphi(p^s)$, p an odd prime, there are precisely $\varphi(\alpha)$ values of x such that exp $x = \alpha (mod\ p^s)$.*‡

Proof: Let g be a primitive root (mod p^s) and set $\alpha d = \varphi(p^s)$. Choose integers r_i such that $r_i \leqq \varphi(p^s)$ and $(r_i, \varphi(p^s)) = d$. The numbers g^{r_i} are by (6) the values of x sought. That there are $\varphi(\alpha)$ distinct values of r_i, and hence by (5) of g^{r_i}, appears thus: every r_i is in the set $d, 2d, \ldots, \alpha d$. Of this set, those in which the coefficient of d is prime to α will have with $\varphi(p^s) = \alpha d$ the highest common factor d. There are $\varphi(\alpha)$ such coefficients.

III. *Analogous theorems mod 2^s.* Evidently 1 is a primitive root (mod 2) and 3 is a primitive root (mod 2^2), but no higher power of 2 possesses a primitive root in the sense defined in (3). Our needs for modulus 2^s, $s > 2$, are met by the following theorems.

(13). *Let τ and s satisfy the relations $s - \tau \geqq 2$, $\tau \geqq 1$; then the necessary and sufficient condition that a is a solution of $a^{2^\tau} \equiv 1 \pmod{2^s}$ is that $a \equiv \pm 1 \pmod{2^{s-\tau}}$.*§

Proof: Evidently a is odd. For $a = 1$, the theorem is obvious. For every odd number greater than 1 there exists a unique value of

*Due to Arndt. Evidently if $s \leqq \mu$, then $\alpha = \exp a \pmod{p^s}$.
†Due to Jacobi and Arndt. See D, pp. 19-20; C, pp. 68-70; M, pp. 27-28.
‡Given in D, p. 17 and C, p. 67 for the special case $s = 1$ and $\alpha = \varphi$.
§Due to Arndt and Bennett.

$\nu \geqq 2$ and a unique choice of sign such that $a = 2^\nu k \pm 1$, with k odd. Then $a^{2^\tau} = (2^\nu k \pm 1)^{2^\tau} = 1 \pm 2^{\tau+\nu} k + \text{terms containing higher powers of 2}$. Thus $a^{2^\tau} \equiv 1 \pmod{2^{\tau+\nu}}$ but $a^{2^\tau} \not\equiv 1 \pmod{2^{\tau+\nu+1}}$. Now if $a \equiv \pm 1 \pmod{2^{s-\tau}}$, then $\nu \geqq s-\tau$ and $a^{2^\tau} \equiv 1 \pmod{2^s}$; while if $a \not\equiv \pm 1 \pmod{2^{s-\tau}}$, $\nu < s-\tau$ and $a^{2^\tau} \not\equiv 1 \pmod{2^s}$.

(14). *Every odd number satisfies the congruence* $a^{2^{s-2}} \equiv 1 \pmod{2^s}$, $s > 2$.

Proof: This is a corollary of (13) with $s - \tau = 2$.

(15). *If* $s > \nu \geqq 2$, *the necessary and sufficient condition for* $\exp a = 2^{s-\nu} \pmod{2^s}$ *is* $a \equiv \pm 1 \pmod{2^\nu}$ *and* $a \not\equiv \pm 1 \pmod{2^{\nu+1}}$.

Proof: Following the argument under (13) with $s = \tau + \nu$ and $s = \tau + \nu + 1$, we have $a^{2^\tau} \equiv 1 \pmod{2^s}$ and $a^{2^{\tau-1}} \not\equiv 1 \pmod{2^s}$, respectively.

(16). *For every s greater than* 2, *exp* $3 = 2^{s-2} \pmod{2^s}$.

Proof: This is a corollary of (15) with $\nu = 2$.

(17). *Every odd integer or its negative is congruent* (*mod* 2^s) *to a power of* 3.

Proof: If $s > 2$, the 2^{s-1} odd numbers $\pm 3, \pm 3^2, \ldots, \pm 3^{2^{s-2}}$ are incongruent each to each (mod 2^s), because $\pm 3^j \equiv \pm 3^k$, $k \leqq j \leqq 2^{s-2}$, implies $3^{j-k} \equiv \pm 1$; this however is impossible since $3^{j-k} \not\equiv 1$, $j \neq k$, by (16) and (4), while $3^{j-k} \not\equiv -1$ because no power of 3 is congruent (mod 8) to -1. The theorem now follows since there are ouly 2^{s-1} incongruent odd numbers. For $s \leqq 2$, the theorem is obvious.

(18). *There are* 2^τ *incongruent numbers x belonging to the exponent* 2^τ (*mod* 2^s) *when* $s - 2 \geqq \tau \geqq 2$; *three numbers when* $\tau = 1$ *and* $s > 2$; *one number when* $\tau = 0$; *one number when* $\tau = 1$ *and* $s = 2$.

Proof: In the first case, by (15) $x \equiv \pm 1 \pmod{2^{s-\tau}}$ and $x \not\equiv \pm 1 \pmod{2^{s-\tau+1}}$. There are $2^{\tau+1}$ values of x admitted by the first of these relations and 2^τ of these values excluded by the second. Hence there are $2^{\tau+1} - 2^\tau = 2^\tau$ numbers in all. In the second case, $x = 2^{s-1} \pm 1$, $2^s - 1$ are the numbers. In the other cases the results are obvious.

IV. *The congruence* $x^a \equiv 1$ *for the general modulus.*

(19). Let $m = 2^s p_1^{s_1} \ldots p_n^{s_n}$, where the p_i are distinct odd primes. We define *Cauchy's maximum indicator*, represented by $\lambda(x)$,* as follows: $\lambda(2^s) = 2^{s-2}$ or 2^{s-1} according as $s > 2$ or $s \leqq 2$; $\lambda(p_i^{s_i}) = \varphi(p_i^{s_i})\, \lambda(m)$ is the lowest common multiple of $\lambda(2^s), \lambda(p_i^{s_i}), \ldots, \lambda(p_n^{s_n})$.

*Symbol due to Carmichael; see C, pp. 52-54. We use Carmichael's proof for (20) below.

(20). $a^{\lambda(m)} \equiv 1 \pmod{m}$ *for all* a *prime to* m.

Proof: $a^{\lambda(2^s)} \equiv 1 \pmod{2^s}$ for $s > 2$, from (14). $a^{\lambda(p_i^{s_i})} \equiv 1 \pmod{p_i^{s_i}}$ and $a^{\lambda(2^s)} \equiv 1 \pmod{2^s}$ for $s = 1, 2$ by the familiar property of the Euler totient. But if, to any modulus, $a^k \equiv 1$, so also is $a^{k\tau} \equiv 1$ for any integer τ. Thus $a^{\lambda(m)} \equiv 1$ to each of the moduli 2^s, $p_i^{s_i}$, and hence to modulus m.

(21). *Let* $d = (\alpha, \lambda(m))$; *then all solutions of either of the congruences* $a^\alpha \equiv 1 \pmod{m}$ *and* $a^d \equiv 1 \pmod{m}$ *satisfy the other.*

Proof: There are known to exist integers u and v such that $u\alpha = d + v\lambda(m)$. Then from $a^\alpha \equiv 1$ we have $a^{\alpha u} \equiv a^{d + v\lambda} \equiv 1$. Since by (20) $a^\lambda \equiv 1$, we must have $a^d \equiv 1$. Conversely from $a^d \equiv 1$ follows $(a^d)^{\alpha/d} \equiv 1$.

(22). *Similarly, if* $a^\alpha \equiv c \pmod{m}$ *then* $a^d \equiv c^u \pmod{m}$, *where* u *is defined in the proof of* (21).

(23). *There are just* d *incongruent solutions of* $x^\alpha \equiv 1 \pmod{p^s}$, *where* $d = (\alpha, \varphi(p^s))$ *and* p *is an odd prime.*

Proof: Let g be a primitive root (mod p^s) and form $k = g^{\varphi(p^s)/d}$. Then the d numbers, $k, k^2, \ldots, k^d \equiv 1$ are the roots of $x^\alpha \equiv 1$, since $(k^t)^\alpha = [g^{\varphi(p^s)}]^{\alpha t/d} \equiv 1$. That these are incongruent follows from (5). That there are only d roots follows by (21) since $x^d \equiv 1$ can have at most d solutions.

(24). *The necessary and sufficient condition that there be numbers* x *belonging* (*mod* m) *to the exponent* α *is that* $\alpha \mid \lambda(m)$.

Proof: The necessity is an immediate consequence of (4) and (20). To prove the sufficiency, let $\alpha \mid \lambda(m)$, let m be given as in (19), and let $d = (\alpha, \lambda(2^s))$, $d_i = (\alpha, \lambda(p_i^{s_i}))$, $i = 1, 2, \ldots, n$. Select $f, f_1, \ldots, f_n$ such that $f \mid d$, $f_i \mid d_i$ and the least common multiple of $f, f_1, \ldots, f_n$ is α.* Since the least common multiple of the d's is α,† this is always possible. Then select a number x such that exp $x = f$ (mod 2^s) and exp $x = f_i$ (mod $p_i^{s_i}$). This is always possible by (6), (11), (18) and (1). That exp $x = \alpha$ follows from (7).

(25). *There are* N *incongruent solutions of* $x^\alpha \equiv 1 \pmod{2^s}$ *where* $N = 2(\alpha, 2^{s-2})$ *whenever* α *is even and* $s \neq 1$; *otherwise* $N = 1$.‡

*Of course we could take $f = d$, $f_i = d_i$. The above choice provides for all possible values of x.

†To show this, let q be a prime such that α is divisible by q^t but not by q^{t+1}. Then $q^t \mid \lambda(m)$, so that q^t is a divisor of at least one of $\lambda(2^s), \lambda(p_1^{s_1}), \ldots, \lambda(p_n^{s_n})$, whose lowest common multiple is $\lambda(m)$. Hence q^t is a divisor of at least one of the d's, and is thus a divisor of D, their lowest common multiple. Since this holds for every q, then $\alpha \mid D$. On the other hand, each d is a divisor of α, so that $D \mid \alpha$. Hence $D = \alpha$.

‡Implied in certain results of Arndt.

Proof: Let $d=(\alpha, \lambda(2^s))=2^r$. By (21) we may replace $x^\alpha \equiv 1 \pmod{2^s}$ by $x^{2^r} \equiv 1 \pmod{2^s}$. If α is even and $s>2$, we have $1 \leqq r \leqq s-2$ and hence by (13) $x \equiv \pm 1 \pmod{2^{s-r}}$. There are 2^{r+1} such numbers less than 2^s. For the cases $s=2$, $s=1$, and the case α odd, the situation is apparent.

(26). Finally, the solutions of $x^\alpha \equiv 1 \pmod{m}$, where m has the form given in (19), are simultaneous solutions of the congruences $x^\alpha \equiv 1 \pmod{2^s}$; $x^\alpha \equiv 1 \pmod{p_i^{s_i}}$, and may be found from the solutions to the separate moduli by use of (1). There are thus $N \cdot d_1 \cdot d_2 \ldots d_n$ solutions in all, where N is described in (25) and $d_i=(\alpha, \varphi(p_i^{s_i}))$.*

V. *The general binomial congruence.*

(27). *A necessary and sufficient condition that* $x^\alpha \equiv c \pmod{p^s}$ *shall have a solution, where* p *is prime and if* $p=2$ *then* $s<3$, *is that* c *be congruent to a power of* g^d, *where* g *is a primitive root* $\pmod{p^s}$ *and* $d=(\alpha, \varphi(p^s))$. The condition may be put in the form $c^{\varphi(p^s)/d} \equiv 1 \pmod{p^s}$.†

Proof: The condition is sufficient, for if $c \equiv (g^d)^t$ and, as in (21), $u\alpha = d + v\varphi(p^s)$, then $x^\alpha \equiv c \equiv (g^d)^t \equiv g^{u\alpha t} \div (g^{st})\varphi(p^s) \equiv g^{u\alpha t}$ has the obvious solution $x \equiv g^{ut}$. But then, by our introductory discussion, it has d solutions since by (23) $y^\alpha \equiv 1$ has d solutions. Conversely by (5) we may put $x \equiv g^\gamma$ and we then have $c \equiv x^\alpha \equiv (g^d)^{\gamma\alpha}/d$. Further, from $c \equiv g^{dt}$ and $(g^t)^{\varphi(p^s)} \equiv 1$, (Fermat's theorem), we obtain easily $c^{\varphi(p^s)/d} \equiv 1$. From $c^{\varphi(p^s)/d} \equiv 1$, since we may put by (5) $c \equiv g^\gamma$, we have $g^{\gamma\varphi(p^s)/d} \equiv 1$; hence by (3) and (4), γ is a multiple of d.

(28). *If there exists a solution for* $x^\alpha \equiv c \pmod{p^{t+1}}$ *where* p *is an odd prime and* $(\alpha, p^{t+1})=p^t$, *there exists a solution also for* $x^\alpha \equiv c \pmod{p^s}$ *for each* $s>t+1$.

Proof (by induction): Let $\alpha = kp^t$, $0 \leqq t < s-1$ and $(k,p)=1$. Assume $x_0^\alpha \equiv c \pmod{p^{s-1}}$ or $x_0^\alpha - c = vp^{s-1}$. We set $x = x_0 + up^{s-t-1}$ and show that u can be so determined that $x^\alpha \equiv c \pmod{p^s}$. $x^\alpha - c$ becomes $(x_0+up^{s-t-1})^\alpha - c = x_0^\alpha - c + \alpha x_0^{\alpha-1}up^{s-t-1} + \alpha(\alpha-1)x_0^{\alpha-2}u^2p^{2s-2t-2}/2 + \ldots = vp^{s-1} + kx_0^{\alpha-1}up^{s-1} + k(\alpha-1)x_0^{\alpha-2}u^2p^{2s-t-2}/2 + \ldots$. Since $2s-t-2 \geqq s$, this gives $x^\alpha - c \equiv vp^{s-1} + kx_0^{\alpha-1}up^{s-1} \equiv p^{s-1}(v + kx_0^{\alpha-1}u) \pmod{p^s}$. This will be congruent to zero $\pmod{p^s}$ if $v + kx_0^{\alpha-1}u \equiv 0 \pmod{p}$. For this linear congruence there is always a unique solution $\pmod{p}$. If $\alpha \neq kp^t$, employ first (22).

(29). *There is always just one solution for the congruence* $x^\alpha \equiv c \pmod{2^s}$ *when* α *is odd.*

*Noted by Lagrange.
†Due to Euler.

Proof: We need merely replace the given congruence by $x=c^u \pmod{2^s}$, according to (22), noting that the converse is evidently true when $d=1$.

(30). *A necessary and sufficient condition that* $x^\alpha \equiv c \pmod{2^s}$ *shall have a solution, for* α *even, is that* c *be congruent to a power of* 3^d, *where* $d=(\alpha, \lambda(2^s))$. The condition may be put in the form, $c\equiv 1 \pmod{4d}$ if $s>2$ but $c\equiv 1 \pmod{2d}$ if $s\leqq 2$.

Proof: Let $s>2$. If $c\equiv(3^d)^t$ and as in (21) $u\alpha=d+v\lambda(2^s)$, then $x^\alpha\equiv c\equiv 3^{dt}\equiv 3^{u\alpha t}\div(3^{vt})^{\lambda(2^s)}\equiv 3^{u\alpha t}$ has the obvious solution $x\equiv 3^{ut}$. It has then $2d$ solutions $3^{ut}y_i$, where y_i are the $2d$ solutions, by (25), of $y^\alpha\equiv 1$. Conversely, any solution of $x^\alpha\equiv c$ may be put, by (17), in the form $x\equiv\pm 3^\gamma$. Then $c\equiv x^\alpha\equiv 3^{\alpha\gamma}\equiv(3^d)^{\gamma\alpha/d}$. Furthermore, $d=\lambda(4d)$, and since $d\mid\lambda(2^s)$ we have $4d\mid 2^s$. Then if $c\equiv 3^d \pmod{2^s}$ we must have $c\equiv 3^d\equiv 1 \pmod{4d}$ by (14). To prove the sufficiency of the condition, we note that $h\equiv 1 \pmod{2j}$ implies $h^2\equiv 1 \pmod{4j}$, and hence from $c\equiv 1 \pmod{4d}$ we have $c^2\equiv 1 \pmod{8d}$, $c^4\equiv 1 \pmod{16d}$, $\ldots$, $c^{2^s/4d}\equiv 1 \pmod{2^s}$. Put now $c\equiv 3^\gamma$ from (17), noticing that if $c\equiv -3^\gamma$ then $c\not\equiv 1 \pmod 8$. Thus we have $c^{2^s/4d}\equiv 3^{\gamma 2^s/4d}\equiv 3^{\gamma\lambda(2^s)/d}\equiv 1$. Hence by (4) and (16), $d\mid\gamma$ and thus $c\equiv 3^\gamma$ is a power of 3^d. The stated theorem, becomes obvious for the cases $s=1, 2$.

(31). Finally the congruence $x^\alpha\equiv c$ has a solution (mod m) provided it has a solution simultaneously to each of the moduli 2^s, $p_1^{s_1}, \ldots, p_n^{s_n}$. The solution may be found directly from (1) when the solutions to the individual moduli are known.

It should be noted in the above that results are obtained in a finite number of steps which can be actually carried out; also that nothing is introduced which lies beyond the reach of a beginner in number theory. This does not mean however, that considerable labor may not be involved if relatively large numbers appear in the congruences.

The Harmonic Mean

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An average which has long been known and which is required in certain problems is the harmonic mean (H. M.). For the n positive values $x_1, x_2, \ldots\ldots x_n$, it is defined as the reciprocal of the arithmetic mean of the reciprocals of the values. In symbols,

$$(1) \qquad \text{H. M.} = \frac{1}{1/n\,(1/x_1 + 1/x_2 + \ldots 1/x_n)} = \frac{n}{\sum(1/x_i)}.$$

This measure is used in averaging ratios, such as rates and prices, when certain conditions are agreed upon.

In the case of time rates, we have ratios between two quantities one of which is in units of time, which we will denote by t, and the other is in units of some element like distance or accomplishment or temperature, etc. Denote this second element, different from time, by d. Then we make the following observations:

(a) A rate may be stated either in the form d/t or in the form t/d. Thus, a car which travels at the rate of 30 miles per hour may also be said to travel at the rate of 2 minutes per mile. In this illustration the second form is not the usual way of expressing the rate, but there are cases in which the form t/d is usual. When we say a man takes 10 seconds to run 100 yards we are expressing his rate in time per unit of distance (t/d).

(b) In averaging rates one should first decide whether d or t should properly be the basic or "fixed" element in the discussion. Occasionally there is a difference of opinion about which element should most appropriately be regarded as fixed. For example, suppose a class of students has been given 15 minutes in which to work as many as they can of a given list of problems, and the number of problems worked correctly by each student recorded. Some educational statisticians would say that *time* should be the fixed element here and that number of problems solved (in a unit of time) should be the variable. Others would say that the number of minutes (t) a student required to work one problem is the proper variable and that a problem (d) should be regarded as the fixed element in the discussion.

In one case the rates are equally weighted in the sense of time and in the other case they are equally weighted in the sense of the element *d*.

(c) The H. M. of the rates expressed in the form d/t gives the same result as the arithmetic mean (A. M.) of the same rates expressed in the form t/d. This is evident from equation (1) if it is written in the form,

$$\frac{1}{H.\,M.}=\frac{1}{n}\sum\frac{1}{x_i}$$

and from the fact that rates in one form are merely the reciprocals of the same rates in the other form.

As an illustration, let us consider three cars:

I { A travels at the rate of 15 miles per hour ($1/4$ mile per minute).
B travels at the rate of 20 miles per hour ($1/3$ mile per minute).
C travels at the rate of 30 miles per hour ($1/2$ mile per minute).

But their rates could just as well have been stated as

II { A travels at the rate of 4 minutes to the mile.
B travels at the rate of 3 minutes to the mile.
C travels at the rate of 2 minutes to the mile.

The H. M. of the rates as stated in I is 20 miles per hour; i. e., $1/3$ of a mile per minute, and the arithmetic mean of the rates as stated in II is 3 minutes per mile or again, 20 miles per hour. (Verify.)

The third observation, i. e., (c) above, suggests the following discussion. The *arithmetic* mean of the rates in I is 21 $2/3$ m. p. h. and this is the H. M. of the rates as stated in II.

The question arises, which is the correct average, 20 m. p. h. or $21\frac{2}{3}$ m. p. h.? The problem is indeterminate until it be agreed whether time or distance is the fixed element. The correct average will differ according to the condition agreed upon. This will be made clear in the following analysis:

Case I. Let

$$x_i=\frac{d_i}{t_i}$$

denote the *i*th rate, $i=1,2,\ldots n$. Then the average rate is

$$\frac{D=\text{ total distance}}{T=\text{total time}}=\frac{t_1x_1+t_2x_2+\ldots+t_nx_n}{t_1+t_2+\ldots+t_n}$$

Condition 1. Let distance be the fixed element, i. e., let d be the constant. Then $d=t_ix_i$, and $t_i=d/x_i$. Therefore, the expression for average rate becomes

$$\frac{\sum t_ix_i}{\sum \frac{d}{x_i}}=\frac{nd}{d\sum \frac{1}{x_i}}=\frac{1}{\frac{1}{n}\sum\frac{1}{x_i}}$$

which is the harmonic mean.

Condition 2. Suppose t is the fixed element. Then $\sum t_ix_i$ becomes $t\sum x_i$ since t is a constant, and $\sum t$ becomes nt. Hence, we have for the average rate,

$$\frac{D}{T}=\frac{t\sum x_i}{nt}=\frac{1}{n}\sum x_i$$

which is the arithmetic mean.

Case II. Let $x_i=\frac{t_i}{d_i}$ denote the ith rate. Then the average rate is

$$\frac{T=\text{ total time}}{D=\text{ total distance}}=\frac{\sum t_i}{\sum d_i}.$$

Condition 1. Suppose d is the fixed element. Then $t_i=dx_i$ and $d=t_i/x_i$. Hence we have

$$\frac{T}{D}=\frac{d\sum x_i}{nd}=\frac{\sum x_i}{n}.$$

Condition 2. Let t be fixed. Then $d_i=\frac{t}{x_i}$ and the average rate is

$$\frac{T}{D}=\frac{nt}{t\sum\frac{1}{x_i}}=\frac{1}{\frac{1}{n}\sum\frac{1}{x_i}}$$

We therefore state the following rules for averaging rates:

Rule 1. The H. M. is used whenever the fixed element is d and the rates are expressed in the form d/t, or when the fixed element is t and the rates are expressed in the form t/d.

Rule 2. The A. M. should be used when the fixed element is t and the rates are expressed in the form d/t, or when the fixed element is d and the rates are expressed in the form t/d.

In the case of prices, which are of course ratios, a similar discussion would apply if the unit of time is replaced by a unit of money. That is to say, prices are ratios between two quantities, one of which is in units of money and the other in units of some commodity or service. They may be stated as so much money per unit of commodity (m/c), or as so many units of commodity per dollar (c/m). Thus, if 100 bushels of wheat are exchanged for 75 dollars of gold, the price of the wheat in terms of the gold is $75 \div 100$, or three-fourths of a dollar of gold per bushel of wheat. Contrariwise, the price of gold in terms of wheat is $100 \div 75$, or one and one-third bushels of wheat per dollar of gold. There are always two such prices in any exchange.

In averaging prices the correct average will depend upon how the prices are stated and upon whether a unit of the commodity (or service) or a unit of money is the fixed element.

The following reference is recommended, "The Nature and Use of the Harmonic Mean", Ferger, *Journal of the American Statistical Association*, Vol. 26, pp. 36-40.

EXAMPLES

1. A does a unit of work in a factory in 4 minutes, B in 5 minutes, C in 6, D in 10, and E in 12. What is their average rate of working?

Here the rates are expressed in the form t/d but it would seem appropriate to regard t as the fixed element because the amount of work done in a unit of time is presumably the important consideration in a factory. So by Rule 1,

$$\text{H. M.} = \frac{5}{1/4+1/5+1/6+1/10+1/12} = \frac{300}{48}$$

$$= 6\tfrac{1}{4} \text{ minutes per unit}$$

$$= \frac{4}{25} \text{ units per minute.}$$

2. A tourist purchases gasoline at three stations, as follows:

Station	*Number of gallons of gasoline for \$1.00*
A	5
B	7
C	6

Here, the prices are given in the form c/m and it would seem appropriate to regard gallon (c) as the fixed element and prices (m) per gallon as the variable quantities which are to be averaged. Hence, replacing d/t by c/m and "rates" by "prices" in Rule 1, we are led to find the harmonic mean.

$$\text{H. M.} = \frac{3}{1/5+1/6+1/7}$$

$$= \frac{630}{107} \text{ gallons per \$1.00}$$

$$= \frac{\$107}{630} \text{ per gallon.}$$

Before I close the consideration of Jacobi, I should like to mention a fact which appears to me not unimportant as regards the viewpoint of the characterization of the man, as well as regards the development of our science. As is well known the year 1812 brought the emancipation of the Jews in Prussia. Jacobi is the first Jewish mathematician who assumes a leading rôle in Germany. Even herein he stands at the peak of a great evolution, important for our science. By this law a new great reservoir of mathematical talent is opened for our land, whose powers soon prove fruitful in our science—beside the extra supply won due to French emigrants. To me a strong invigoration of science seems to be won by such a sort of renewal of blood; beside the above mentioned law of migration of productivity I should like to designate the appearance of this phenomenon as the effect of national "infiltration."—Felix Klein, *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert* (1926), pp. 114-15.

Invariants and Elementary Mathematics

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1. *Introduction.* To say that the concept invariant is one of the cardinal notions of modern mathematics is to express a mathematical commonplace. It was some time ago, as a matter of fact, that Lie made his oft quoted statement: "In recent times the idea becomes more and more prevalent that many branches of mathematics are nothing but the theory of invariants of special groups". The best known illustration of Lie's statement is furnished by Euclidean geometry and yet of the nine texts treating analytic geometry that have found their way into the writer's library only one contains the word *invariant* in its index.

The purpose of the present note is to call attention to the fact that some of the extremely simple notions of invariant theory may be employed to advantage in undergraduate teaching. More specifically, we shall point out that in certain geometrical demonstrations we may (provided we are *invariant conscious*) confine our attention without loss of generality to very simple figures. In the writer's opinion one of the objectionable features of elementary mathematics, as presented in the traditional manner, is the practice of demonstrating general theorems by means of figures and ignoring the question of the generality of the figures. A considerable number of elementary theorems are immediate consequences of the distributive property of the dot and cross products of vector analysis, but unfortunately, the usual demonstrations of these distributive laws are not pleasing from the standpoint of the generality of figures. In article 3 we use vector methods but avoid the difficulty just mentioned by invariantive considerations.

We shall assume the notion of Cartesian coordinate system and the fact that a linear equation in a plane Cartesian coordinate system represents a straight line.

Perhaps the simplest notion of an invariantive nature is that of *invariant expression.* By an invariant expression under a transformation x,y to $\bar{x},\bar{y}$, we mean a quantity which has the same numerical value and the same functional form in each of the two systems x,y and $\bar{x},\bar{y}$. For example, x^2+y^2 is such an invariant under rotation of axes. In general, any object of thought which does not change when we make a certain transformation of coordinates will be said to be an

invariant under that transformation. We regard a transformation of coordinates as attaching a new set of numbers (the new coordinates) to each point but as leaving the point fixed—that is each point of our space is an invariant under all coordinate transformations.

2. *Simple invariants under translation.* The simplest non-trivial coordinate transformation is that associated with a translation of axes, namely:

$$\bar{x}=x+h;\quad \bar{y}=y+k. \tag{2.1}$$

Because of its extreme simplicity its invariants are numerous. Among the obvious ones which are encountered in elementary mathematics are: the points and lines of the given plane, the slope of a line, the distance between two points, the intersection angles of two lines, and the expressions x_2-x_1, y_2-y_1 (x_1,y_1 and x_2,y_2 being the coordinates of two fixed points). Some almost trivial but nevertheless worthwhile applications of the foregoing observations may be made to the question of the generality of the figures involved in certain demonstrations in analytic geometry. For example, in proving the formula

$$d^2=(x_2-x_1)^2+(y_2-y_1)^2$$

most authors use two figures, one having the two points lying in the first quadrant and the other having one point in one quadrant and the other point in another quadrant. One writer, at least, is good enough to state explicitly that his demonstration establishes the theorem only in the cases covered by his figures and that the student should investigate the other cases. Evidently there are ten ways in which the two points may be distributed with respect to the quadrants (neither point 0,0) any one of which may be converted into any other by translation of axes. Such a translation leaves both d^2 and $(x_2-x_1)^2+(y_2-y_1)^2$ unchanged and therefore (except for the matter of the direction of the segment joining the points) it suffices to consider only one case say both points in the first quadrant. One could take one point at 0,0 and note that the theorem holds regardless of the direction in which the second point lies. Similar remarks apply to the formula for the slope of the line joining two points. Also, a slight gain in simplicity may be obtained in the proof of the formula for the tangent of the angle of intersection of two lines by drawing the $x-$axis through the point of intersection.

3. *An invariant under a homogeneous linear transformation.* We have assumed that a linear equation in Cartesian coordinates represents a straight line and that the system x,y is rectangular Cartesian.

Consequently, the coordinate curves $\bar{x}=a$ constant, $\bar{y}=a$ constant, determined by the transformation

$$\bar{x}=cx+dy; \quad \bar{y}=ex+fy; \quad cf-de\neq 0, \tag{3.1}$$

are straight lines.

We shall take as the unit of measure for distance in all coordinate systems the distance between the points 0,0 and 1,0 in the x,y system and require further that the coefficients in (3.1) be such that the square of the distance from the origin to any point be given by the expression x^2+y^2, i. e., we require that the sum of the squares of the coordinates be invariant under (3.1). Upon substituting (3.1) into $\bar{y}^2+\bar{x}^2$ and equating the result identically to x^2+y^2, we see that the necessary and sufficient condition for this invariance is

$$c^2+e^2=d^2+f^2=1; \qquad cd+ef=0. \tag{3.2}$$

Now if x_1,y_1 and x_2,y_2 are the coordinates of two points P_1 and P_2 then the quantities x_2-x_1 and y_2-y_1 are called the x and y components of the vector $\overrightarrow{P_1P_2}$. Similarly, the difference of the barred coordinates of P_1 and P_2, namely, $\bar{x}_2-\bar{x}_1$ and $\bar{y}_2-\bar{y}_1$ will be called the components of the vector $\overrightarrow{P_1P_2}$ in the $\bar{x},\bar{y}$ system. Evidently, the vector itself is an invariant while its components follow the transformation law (3.1). Consequently, we have

$$(\bar{x}_2-\bar{x}_1)^2+(\bar{y}_2-\bar{y}_1)^2=(x_2-x_1)^2+(y_2-y_1)^2. \tag{3.3}$$

A natural generalization of the left member of (3.3) can be obtained by replacing the squares of the components with the products of corresponding components of two different vectors and an examination of the derivation of (3.2) shows that the quantity so obtained is likewise an invariant. Thus we have the theorem:

Theorem (3.1). *If a_1,a_2 and b_1,b_2 are the components of two vectors α and β then the expression $a_1b_1+a_2b_2$ is an invariant under the transformation defined by* (3.1) *and* (3.2) i. e.,

$$\bar{a}_1\bar{b}_1+\bar{a}_2\bar{b}_2=a_1b_1+a_2b_2. \tag{3.4}$$

In order to find a meaning for this invariant, we shall endeavor to so select the $\bar{x},\bar{y}$ system that $\bar{a}_1=a=[(a_1)^2+(a_2)^2]^{\frac{1}{2}}$, $\bar{a}_2=0$, since this will reduce our invariant to the product of the invariant a and $\bar{b}_1$ and it will suffice to express $\bar{b}_1$ in terms of invariants. By inspection we see that our end will be accomplished if we take $c=a_1/a$; $d=a_2/a$;

$e=-a_2/a$; $f=a_1/a$. Furthermore, it is allowable to do this since (3.2) will be satisfied and $cf-de$, the determinant of our transformation, will be unity. We note in passing that if Θ is the angle between ox and α then $a_1/a=\cos\Theta$, $a_2/a=\sin\Theta$. It will be recalled that the loci obtained by holding $\bar{x}$ constant (or $\bar{y}$ constant) in (3.1) are straight lines. The slopes of these lines in the special case with which we are now dealing, are $-a_1/a_2$ and a_2/a_1. In other words the $\bar{x}$ and $\bar{y}$ lines are orthogonal. Furthermore, if the origin of α is placed at 0,0, the $\bar{x},\bar{y}$ coordinates of the vertex of α will be a,0 and the $\bar{x},\bar{y}$ coordinates of $\bar{x}/a\cdot\alpha$ (a vector of magnitude $|\bar{x}|$) will be $\bar{x}$,0. That is $\bar{x}$ represents distance measured on the $\bar{x}-$axis. Thus we see that the $\bar{x},\bar{y}$ system is rectangular Cartesian and that α has the direction of the positive $\bar{x}-$axis. Consequently, b_1 is the product of the invariants b ($b=|\beta|$) and the cosine of the angle between α and β. Thus,

$$\alpha\cdot\beta=a_1b_1+a_2b_2=ab\cos(\alpha,\beta). \tag{3.5}$$

A most important property of the dot product (usually proved by means of a figure) is expressed by the equation

$$\alpha\cdot(\beta+\gamma)=\alpha\cdot\beta+\alpha\cdot\gamma \quad \text{(the distinctive law)}. \tag{3.6}$$

We shall confine ourselves at present to the case in which α, β, γ are coplanar.

Proof: If b_1,b_2 and c_1,c_2 are the x,y components of β and γ, then, by definition of the sum of two vectors, the components of $\beta+\gamma$ are b_1+c_1, b_2+c_2, and consequently, we may write

$$\begin{aligned}\alpha\cdot(\beta+\gamma)&=a_1(b_1+c_1)+a_2(b_2+c_2)\\&=a_1b_1+a_2b_2+a_1c_1+a_2c_2=\alpha\cdot\beta+\alpha\cdot\gamma.\end{aligned} \tag{3.7}$$

As an application of (3.5) and (3.6) we shall (following a well known procedure) establish two theorems of trigonometry* and as a corrollary of one of them establish (3.6).

The Law of Cosines. Referring to figure (3.1) we have at once:

$$c^2=(\alpha+\beta)\cdot(\alpha+\beta)=a^2+b^2+2ab\cos(\alpha,\beta). \tag{3.8}$$

By means of the law of cosines, we may now establish the non-coplanar case of (3.6). Thus let a_1,a_2,a_3; b_1,b_2,b_3 be the x,y,z components of α and β in a 3-dimensional rectangular Cartesian coordinate system, the x,y plane of which does not necessarily coincide with

*For a different vectorial proof of the addition formulas of trigonometry see, W. E. Byrne, *Vector analysis and trigonametry* National Mathematics Magazine, February, 1935, p. 130.

the α,β plane. Referring to figure (3.2) and applying the law of cosines, or the distributive law, we obtain at once

$$(3.9) \qquad |\alpha-\beta|^2=(a_1-b_1)^2+(a_2-b_2)^2+(a_3-b_3)^2=a^2+b^2-2\alpha\cdot\beta.$$

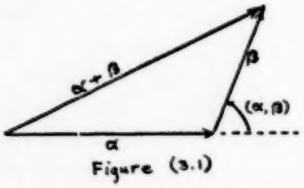

Figure (3.1)

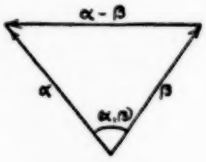

Figure (3.2)

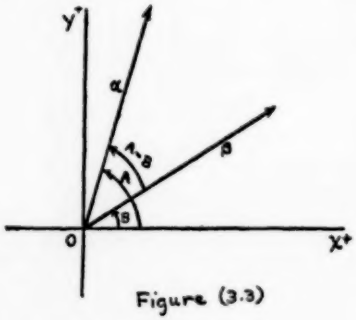

Figure (3.3)

Here, we have used the facts that the components of $\alpha-\beta$ are a_1-b_1, a_2-b_2, a_3-b_3 and that the square of the diagonal of a rectangular box is the sum of the squares of the sides to establish the first equality, and of course the law of cosines to establish the second. Obviously, (3.9) reduces to

$$(3.10) \qquad \alpha\cdot\beta=a_1b_1+a_2b_2+a_3b_3$$

and the distributive law for the non-coplanar case follows.

The Formula for cos$(A-B)$. Let α and β of figure (3.3) be two unit vectors making angles A and B, respectively with ox, then whatever the magnitudes of A and B, the components of α and β are cos A, sin A and cos B, sin B, respectively. Consequently, by virtue of (3.5) we have at once

$$(3.11) \qquad \alpha\cdot\beta=1\cdot1\cdot\cos(A-B)=\cos A\cos B+\sin A\sin B.$$

If A is positive and B negative (3.11) gives us the formula for cos $(A+B)$. Apparently, this proof is not only simpler than the traditional proof given in the trigonometries but its nature makes the discussion of generality easy.

The applications of the dot product to geometry and physics are quite numerous. As a final example, let us consider the problem of finding the relationship between two sets of rectangular Cartesian coordinates with common origin. Thus let us suppose that we have two such systems given and let $i,j,k,\bar{i},\bar{j},\bar{k}$ designate unit vectors along the $x,y,z,\bar{x},\bar{y},\bar{z}$ axes respectively. Then ρ, the radius vector from the origin to the point $P(x,y,z;\bar{x},\bar{y},\bar{z})$, is given by

$$\rho = xi+yj+zk = \bar{x}\bar{i}+\bar{y}\bar{j}+zk. \tag{3.12}$$

Multiplication of (3.12) by $i\cdot$ yields one of the desired relationships

$$x = i\cdot\bar{x}\bar{i}+i\cdot\bar{y}\bar{j}+i\cdot\bar{z}\bar{k}. \tag{3.13}$$

The others may be found by multiplying by $j\cdot, k\cdot, \bar{i}\cdot$, etc.

In the special case of rotation of Cartesian axes in the x,y plane we may delete the z-terms from (3.12) and proceed as before. Thus to find expressions for $\bar{x}$ and $\bar{y}$, we multiply by $\bar{i}\cdot$ and $\bar{j}\cdot$. The result is

$$\bar{x} = \bar{i}\cdot ix+\bar{i}\cdot jy; \quad \bar{y} = \bar{j}\cdot ix+\bar{j}\cdot jy. \tag{3.14}$$

4. *An invariant under rotation.* Upon examining the appropriate figures, we see that equations (3.14) may be put in the familiar form

$$\bar{x} = x\cos\theta+y\sin\theta; \quad \bar{y} = x(-\sin\theta)+y\cos\theta. \tag{4.1}$$

An invariant under this transformation which involves the sine of the angle between two vectors in the given plane is the determinant of their components. Since the equations (4.1) are linear they induce the similar transformation

$$\bar{a}_1 = a_1\cos\theta+a_2\sin\theta, \quad \bar{a}_2 = a_1(-\sin\theta)+a_2\cos\theta \tag{4.2}$$

in the components of any given vector α in the plane. Now substitution of (4.2) in the expression $a_1b_2-a_2b_1$ yields the equality

$$\bar{a}_1\bar{b}_2-\bar{a}_2\bar{b}_1 = a_1b_2-a_2b_1 \tag{4.3}$$

and thus we have the following theorem:

Theorem (4.1). The determinant of the components of any two vectors lying in a given plane is invariant under rotation of rectangular Cartesian axes.

To obtain a meaning for this invariant, we so rotate the axes that $\bar{a}_1 = a$ (a is the magnitude of α) $\bar{a}_2 = 0$. Our invariant then reduces, in the $\bar{x},\bar{y}$ system, to $a\bar{b}_2$. We note that $\bar{b}_2$ is equal to the magnitude

of β times the quantity $\sin(\alpha,\beta)$, if $\sin(\alpha,\beta)$ is taken to be positive or negative according as the rotation α into β has the same or opposite sense as the rotation *ox* into *oy*. Hence we may assert

Theorem (4.2). ***The invariant $a_1b_2-a_2b_1$ is equal to the product of the magnitude of α, the magnitude of β, and the sine of the angle between α and β.***

Evidently, this expression also is distributive, i. e., $a_1(b_2+c_2)-a_2(b_1+c_1)=(a_1b_2-a_2b_1)+(a_1c_2-a_2c_1)$. Consequently, it may be regarded as a kind of product of two vectors, as a matter of fact, it is the magnitude of the so-called "cross product" of α and β, $\alpha\times\beta$.

An immediate corrollary of theorem (4.2) is the formula for $\sin(A-B)$. For referring to figure (3.3), we have at once

$$\sin(A-B)=\sin A\cos B-\cos A\sin B. \tag{4.4}$$

5. *Conclusion.* The writer is of course aware that many of the questions of generality involved in elementary mathematics may be disposed of without the use of invariants. But the point is that usually the question is not raised at all and if it is disposed of by elementary means an excellent opportunity is lost to employ one of the outstanding notions of modern mathematics—a notion which should not be confined to a single application to conic sections.

That there is considerable room for modernization of elementary mathematics* has been pointed out before in this journal. The writer believes that invariants and vector analysis should play conspicuous roles in geometry and trigonometry. In algebra—determinants should be developed through the *e*-systems of tensor analysis. This would not only provide an excellent introduction to the technique of tensor analysis but would provide the background for the definition of the cross product in terms of the three index *e*-symbols.

*W. E. Byrne, *loc. cit.*

Humanism and History of Mathematics

Edited by
G. WALDO DUNNINGTON

Unpublished Mathematical Manuscripts in American Libraries

By A. W. RICHESON
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In searching for an early American mathematical manuscript supposedly written during the first quarter of the 18th century, the attention of the writer was directed to other unpublished manuscripts written prior to 1800 and now deposited in American libraries.

In attempting to collect this information a number of difficulties were encountered. First, it is impossible for one person to examine all the libraries and museums throughout the country in a short space of time; second, many of the collections are very poorly catalogued and for this reason alone complete lists of their manuscripts are difficult to obtain; third, after the lists of manuscripts have been obtained it is impossible to state definitely, in many cases, that a given manuscript has not been published at some time. The procedure used in collecting the information for this paper was as follows: about 250 of the largest college, university, private, state, and municipal libraries throughout the country were requested to give a list, with as much information as possible, of their mathematical manuscripts which were written prior to 1800. These were then carefully checked to determine whether or not they had been published at some time.

Eighteen libraries reported 72 unpublished manuscripts. If we include those manuscripts in the Plimpton Collection, we would then have a total of about 175.

The list is as follows:

American Antiquarian Society Library.

AUTHOR UNKNOWN. *A Plan of Learning.* Done by a student at Portsmouth University. A treatise on mathematics from arithmetic to gunnery and fortification, including geometry, trigonometry, spherics, astronomy, navigation, and marine surveying. Ms., paper. *Ca.* 1777.

GREEN, JOSHUA. *His Book.* Ms., paper, 66 pp. arithmetic and 6 pp. geometry. 1782.

HALE, ROBERT. *Arithmetic.* Containing treatises on arithmetic, logic, metaphysics, physics, geography, and geometry. Ms., paper, 202 pp. 1720.

LAWTON, CHRISTO JACOB. *His Book.* Ms., paper. 1732.

WATERS, EBENEZER. *His Ciphering Book.* Ms., paper, 5 pp. decimals, 8 pp. geometrical constructions, and 6 pp. trigonometry. Dated March 4, 1760.

Library of the Boston Athenæum.

AUTHOR UNKNOWN. *The schoolmaster's companion, and young man's guide to vulgar and decimal arithmetic, mensuration, gauging,* etc. . . . *By a friend to improvement.* "The author considers his work a supplement to Pike's *Arithmetic.*" Ms., paper. *Ca.* 1800.

Boston Public Library.

BOWDITCH, NATHANIEL. *The Compleat Measurer* [Astronomy]. Ms., paper, 229 pp., folio. Diagrams. Salem, 178-.

——, ——. *Notebook on Navigation.* Ms., paper, 253 pp., folio. Diagrams. Salem. 1786.

——, ——. *The Practical Surveyor and General Gauger.* Ms., paper, 120 pp., folio. Diagrams. Salem, 1787.

——, ——. *Notebook of Algebra.* Ms., paper, 323 pp., 4to. Salem, 1788.

——, ——. *Manuscript of commonplace books relating to general mathematical subjects.* Ms., paper, 22 vols., 4to and folio. Salem, 1791-181—.

Connecticut Historical Society Library.

BRIDGHMAN, MARY. *Arithmetic manuscript kept by her.* Ms., paper. Date uncertain but probably early 18th-century.

Chicago Historical Society Library.

MORTON, JOHN. *A mathematical exercise book, which contains accounts for settlement of estates in 1762 and 1771.* Ms., paper.

Harvard University Library.

DINAN, JAMES. *His Book, Algebra.* Ms., paper, 4 unnumbered pp. +125 numbered pp. 1730.*

LANGDON, SAMUEL. *His Book, Algebra.* Ms., paper, 2 unnumbered pp. +75 numbered pp. +18 unnumbered pp. 1739.*

Henry Huntington Library.

BRICE, ARTHUR. *Practical Geometry by Arthur Brice of the Prince of Wales Royal Military Academy.* Ms., 47 pp., 8vo. Diagrams, plans and maps in color and ink. England. *Ca.* 1784.

Laid in, two maps: I. . . . *Plan of Topography of forty square miles with a Brigade Encamped* . . . . finished by A. Brice November 18, 1784. II *The Attack and Defeat*

*For a detailed description of these two manuscripts, see Simons, Lao G. *Introduction of Algebra into American Schools in the Eighteenth Century,* U. S. Bureau of Education Bulletin, No. 18, pp. 3-17.

of the American Fleet under Benedict Arnold . . . . upon Lake Champlaine the 11th October, 1776.

AUTHOR UNKNOWN. *Geometry.* First Book, Propositions I-XLVIII. Ms., 30 pp., folio. Diagrams. Virginia, *Ca.* 1775.

HASTING PAPERS. *Commonplace book of mathematics and astronomy containing problems, etc., from Euclid and Speidell.* Ms., 77 pp., 4to. In Latin and English. Secretary hand. Diagrams. England, 1623.

AUTHOR UNKNOWN. *An historical account of the arts and sciences of the Indians of Mexico, including a compendium of the history of New Spain before and after the conquest.* Ms., 149 pp., 4to. In Spanish. One table, one diagram. *Ca.* 1775.

The first three chapters, 34 pp., devoted to mathematics. Contents:

I *Sobra la falta de peso los Mexicanos, y modo como lo suplican con sus medidas, y numeros.*
II *Fundamentos, y Operationes de la Arithmetica Mexicana.*
III *Geometria, Hydraulica, Mechanica, y Geografia de los Mexic[ano]s.*

PTOLEMY. *Almagest.* Ms., vellum, 274 leaves, 34 lines to the page. Marginal diagrams. Tables at the end. In 34 dictones. France, 1274. One small historiated initial, other initials in blue, gold, and magenta with quarter and half bar borders, every page ornamented with red and blue pen work.

SPEIDELL, JOHN. *The Use of the Mathematical Scale.* Invented by Jo: Speidell teacher of the Mathematicks in London, contained in Six Books. Ms., 140 pp., folio. Secretary hand. Diagrams. England, *Ca.* 1615. Contents:

I Addition, Subtraction, Multiplication, Division, the rule of pro:[portion], & the Extraction of Roots.
II The Dividinge of all manner of lines . . . . .
III The Reducing of any figure into a Square . . . . .
IV The Cuttinge of from any figure any part or partes Required.
V The Doctrine of Triangles . . . . .
VI The Makinge of Certaine Sunne Dialls.

——, ——. *A Mathematical Treatise in three parts.* Ms., 107 pp., folio. Italic hand. Diagrams. England, *Ca.* 1612.

University of Illinois Library.

AUTHOR UNKNOWN. *Mathematics and its Applications.* Ms., paper, 291 pp. Diagrams and maps. England, second half of the 18th century.*

AUTHOR UNKNOWN. *Mathematical Exercises and Formulæ.* Ms., paper, 144 pp. Diagrams. England, second half of the 17th century.†

GLASER, KONRAD. *Rechenbuchlein.* An elementary commercial arithmetic. Ms., vellum, 10 pp. In German. Konrad Glaser Rechenmeister zu Nurnberg, 1534.‡

ENGELHARD, VALENTIN. *Institutiones Mechanicæ Oder Practica Geometriæ.* Ms., paper, 381 pp. Diagrams. Germany, *Ca.* 1700. Probably a copy from the author's autograph.

*For a detailed description of this manuscript, see Richeson, A. W. NATIONAL MATHEMATICS MAGAZINE, Vol. XI, 1937, pp. 221-230.

†For a detailed description of this manuscript, see Richeson, A. W. *Ibid.*, pp. 165-171.

‡For a detailed description of this manuscript, see Richeson, A. W. *Isis*, Vol. XXVIII, 1938, pp. 341-348.

Maine Historical Society Library:

LARKIN, NEWTON. *A manuscript record book containing arithmetic, algebra, geometry, surveying, trigonometry, and astronomy.* Ms., paper, 589 pp. Diagrams. 1789.

TAYLOR, EPHRAIN. *Arithmetic.* Ms., paper, 1777. Ephrain Taylor was a Scotch schoolmaster in General Washington's army stationed at West Point. He agreed to teach four others arithmetic, if they would take his place on duty.

University of Michigan Library.

List of Greek Papyri:

I *Fragments of a mathematical treatise, with diagrams.* 5x8½ in. Second century A. D.

II *Table of Fractions,* 1/8 to 1/18. 3 ft.6½ in. x 3¾ in. Fourth century A.D.

III *Geometrical text* [?]. 7½x2¼ in. Third century A. D.

IV *Mathematical fragment.* Diagrams, 2 3/8x2¼ in.

V *Table of fractions.* 1¾x1¾ in.

VI *Literary scrap; two columns of prose work, probably geometrical.* 3x2¾ in. Probably first century A. D.

VII *Fragment of an obscure text.* May be mathematical symbols. 2¼x3 in.

VIII *Mathematical problems.* Fragment. Early second century A. D.

IX *Many fragments from a papyrus containing multiplication tables.* Third and and fourth centuries A. D.

List of Arabic Manuscripts:

I *Khulāsat al-hisāb,* a compendium of arithmetic by Bahā al-Dīn al-'Āmilī. Copied A. H. 1127 (1749 A. D.).

II *Risālah fi 'l-hisāb,* a manual of arithmetic.

III A manuscript containing (1) *al-Mulakhkhas,* a work on astronomy by Mahmūd al-Jaghminī, and (2) *Ashkāl al-Ta'sīs,* a work on geometry by al-Samarkandī. Copied A. H. 878 (1500 A. D.).

IV Collection of ten essays by different authors, including *Ashkāl al-Ta'sīs,* the prolegomena of Euclid with an introduction. Copied A. H. 1095 (1717 A. D.).

New York Historical Society Library.

TOWNSEND. *Trigonometry.* A notebook containing definitions and problems in trigonometry. Ms., paper, 75 pp. *Ca.* 1769.

RICHARDS, CAPT. STEPHEN. An account book from 1732 to 1744 which contains at the end 117 pp. of geometrical problems and axioms, problems in trigonometry, 15 problems in calculating dates in the Julian Calendar, problems in sailing and navigation, etc. Ms., paper.

New York Public Library.

LAGNY, M. DE. *Elemens metaphysiques de la science des nombres.* [Par] Mm. de Lagny de L'Academie Royale des Sciences. Pour masdemoiselles [sic] de Pontchartain. Ms., paper, 16 pp., 4to. 1727.

REGNAULT, NOEL, (S. J.) *Cours et Traite de Mathematique donne par le Regnault au Collège-de-Louis-le-Grande, a Paris l'an de Grace 1735.* Ms., paper, 176 pp., 4to.

Diagrams. 1735. Entry on title page, Jacobus Mathon in Physica Mathematica que auditor, 1735.

ROOS, JOHN and JACOB. Textbook in mathematics, including arithmetic, geometry, trigonometry, surveying, navigation, etc. Ms., paper, in four parts. 1783-1793.

VELLA, LORENZO. *Compendio d'arithmetica prattica numerica semplica. Breve trattato della trigonometrica. Breve compendio di geometria prattica. Trattato dell' artiglieria, in cui si dimostra la scola delle parti stigli, proporzioni, ed ornamenti delle cannoni, che sono in servizzio della marina, per le navi della sagra religione di Malta, alla moda, ed uso della marina, di Francia.* Ms., paper, 141 pp., folio. Colored drawings. The latest date given in the manuscript is 1794

AUTHOR UNKNOWN. *A Treatise on Mathematics.* Geography, navigation, astronomy, marine surveying, etc. Ms., 509 pp., folio. Maps and drawings. England, 1799. This manuscript is signed T. L.

University of North Carolina Library.

JONES, EDMUND. *Ciphering Book. Arithmetic.* Ms., paper, 208 pp., 8vo. A revolutionary poem covers the first two pages.

MARSH, DANIEL G. *Ciphering Book.* Ms., paper, 80 pp. folio. *Ca.* 1790.

PETTIGREW, EBENEZER. *Ciphering Book.* Ms., paper, 82 pp., folio. 1792. Definitions and rules are given at the end of each page, examples underneath.

POWELL, MARK. *Ciphering Book.* Ms., paper, 146 pp., folio. 1791.

RYAN, MARTHA. *Ciphering Book.* Ms., paper, 148 pp., folio. 1781. There are beautiful decorative designs and fancy lettering at the heading on each page. The revolutionary influence is shown by mottoes, ship designs, and other patriotic decorations with "Liberty" or "George Washington" written on many of the pages.

Historical Society of Pennsylvania Library.

ALLEN, JAMES. *Conic Sections.* A notebook on conic sections. Ms., paper. No date.

BUFFINGTON, JOHN. *Arithmetic and shoemakers account book.* Ms., paper. 1771-1773.

BUFFINGTON, RICHARD. *Arithmetic and farm book.* Ms., paper. 1770-1776.

GRIFFITH, ISAAC. *Plane Trigonometry.* Ms., paper. 1734.

HILDAS, MARTIN. *Arithmetic.* Ms., paper, 1764.

INKSON, ROBERT. *Arithmetic.* Ms., paper. 1791.

MARSHALL, HUMPHREY. *Algebra.* Ms., paper. No date.

NORRIS, CHARLES. *Arithmetic.* Ms., paper. 1729.

NORRIS, ISAAC. *Arithmetic.* Ms., paper. There are three of these dated 1774, 1775, and 1776.

NORRIS, JOSEPH PARKER. *Notebook on arithmetic, geometry, and trigonometry.* Ms., paper. 1776.

AUTHOR UNKNOWN. Notebook. Containing geometry, trigonometry, and sailing. Ms., paper, loose leaves. 1726-1727.

AUTHOR UNKNOWN. *A compleat system of practical arithmetic. Part of the mathematicks; or the school-master's assistant.* Ms., paper. 1763. Pretace signed William Collis, printed slip pasted on the front cover, Robert Bailey's Book.

AUTHOR UNKNOWN. *A small book on fractions.* Ms., paper. No date.

PATTERSON, ROBERT. *Notebook of Samuel Hayes. Navigation by Robert Patterson.* Ms., paper. 1789.*

BROOKE, ROBERT. *His Book. Arithmetic.* Ms., paper, 180 pp. *Ca.* 1783.†

University of Pennsylvania Library.

PATTERSON, ROBERT. *Notebook. Mathematica Compendia: or a Short System of Mathematical Literature.* Ms., paper, 2 Vols., Vol. I, 170 pp. *Ca.* 1788.‡

The George A. Plimpton Collection.

Mention should be made of the private collection of Mr. George A. Plimpton which contains over one hundred mathematical manuscripts along with numerous first editions and rare copies of mathematical texts. For a description of some of these, see Mr. Plimpton's paper: *The History of Elementary Mathematics in the Plimpton Library.* Science, Vol. LXVIII, pp. 390-395.

Rhode Island Historical Society Library.

BROWN, JOHN. *Ciphering Book.* Ms., paper. Dates of 1749 and 1751 occur in the text.

CARDER. *The Carder family ciphering and account books.* Ms., paper, 8 Vols. Dated from 1700 to 1850.

AUTHOR UNKNOWN. *General mathematical subjects.* Ms., paper. 1792.

St. John's College (Maryland) Library.

AUTHOR UNKNOWN. *Lectures on natural philosophy and conic sections.* Ms., paper. *Ca.* 1800. The handwriting seems to be that of President McDowell, who was professor of mathematics until 1806. He then went to the University of Pennsylvania.

University of Virginia Library.

LATANE, HENRY, *Arithmetic exercises.* Ms., paper.

*For a detailed description of this manuscript, see Simons, Lao G. *Introduction of Algebra into American Schools in the Eighteenth Century,* U. S. Bureau of Education Bulletin, No. 18, pp. 21-26.

†*Ibid.*, pp. 25-26.

‡*Ibid.*, pp. 21-24.

The Teacher's Department

Edited by
JOSEPH SEIDLIN and JAMES MCGIFFERT

The Unification of Plane and Solid Analytic Geometry

By C. R. WYLIE, JR.
Ohio State University

It is the purpose of this note to indicate a device which the author has found helpful in making the transition from plane to solid analytic geometry smoother and easier for his students. It is a simple one, yet the fact that it is neither employed nor even suggested in any text with which the writer is familiar seems to warrant this mention of it.

The difficulties which students so frequently encounter in their first work in solid analytic geometry seem to the author to spring from the dissymmetry of the treatments of the two and three dimensional cases. In the *plane* (in undoubted preparation for later work in calculus, where the tangent of an angle,

$$\frac{dy}{dx},$$

is fundamental) a single direction angle is employed, and direction is almost always specified by giving the tangent of this angle, that is by the single quantity, slope. In *three dimensions*, however, a different, more general, and entirely symmetric scheme is perforce adopted. Three direction angles are used, and direction is specified by giving the cosines of these angles, the cosines of course being connected by a single identical relation.

The differences in the development of plane and solid geometry on the basis of these initial conventions are of course not fundamental, for plane geometry can be developed just as easily and as quickly by using two direction angles and their cosines, as by using one direction angle and its tangent. And the author has found that his students have much less trouble with the three dimensional case after they have carried out this entirely analogous development in two dimensions.

The suggested approach in the plane (which supplements and of course does not supersede the use of the tangent, or slope, on which so much of the calculus is based) leads to a series of results differing from those valid for three dimensions only in that fewer terms are involved. For direction angles we take the angles α and β which a line makes with the positive X and Y axes. The cosines of these angles are the direction cosines l and m. For the line joining $P_1 \equiv (x_1, y_1)$ and $P_2 \equiv (x_2, y_2)$ we then have

$$l = \frac{x_2 - x_1}{P_1P_2} \quad \text{and} \quad m = \frac{y_2 - y_1}{P_1P_2}.$$

Evidently $l^2 + m^2 = 1$. Numbers proportional to l and m are direction numbers. The angle between two lines is given by the formula $\cos \theta = l_1l_2 + m_1m_2$, from which the conditions that two lines be parallel or perpendicular are respectively

$$l_1l_2 + m_1m_2 = 1 \quad \text{and} \quad l_1l_2 + m_1m_2 = 0.$$

The treatment of the straight line in the plane is rather interesting when direction cosines are used, because of the ambiguous role played by the line in two dimensions. The line is surely a linear one-dimensional locus, and as such admits of a treatment similar to that used for the line in three dimensions. The line is also a linear $(n-1)$-dimensional locus (for $n = 2$), and as such must admit of a treatment similar to that used for the plane in three dimensions. If we denote the equations of a line arising from these respective interpretations as an L-form and a P-form (to suggest line and plane respectively) we have

L-form: $$\frac{x - x_1}{l} = \frac{y - y_1}{m}$$

where (x_1, y_1) is a fixed point on the line and l and m are the direction cosines of the line itself.

P-form: $$lx + my = p$$

where l and m are the direction cosines of the normal to the line, and p is the distance from the origin to the line. From the L-form, the point-slope and two-point forms of the equation of a line are immediately obtainable; while the P-form is precisely the familiar normal form of the equation of a line in which instead of using the sine and cosine of the direction angle of the normal we replace the sine of this angle by the cosine of its complement and use two cosines. All the problems of three dimensions involving the relations of lines and planes have

their counterparts in the plane in problems involving the L- and P-forms of the equations of straight lines only.

Most of these results can be developed by the students themselves as exercises, and thus require little or no extra time for their presentation. In the author's experience this procedure is well worth while for the light which it throws on the fundamental similarity of plane and solid analytic geometry, a symmetry hardly apparent in the traditional treatment.

A Vanderbilt freshman pointed out in the author's class that the fraction 2/3 could be changed to its equal in twelfths by getting the difference 12 minus 3 which is 9 and adding 2/3 of 9 or 6 to the numerator 2. Thus the equivalent fraction is found to be 8/12. The method illustrated here is stated in the following rule:

To change a fraction to its equivalent with increased denominator, increase the numerator by the fraction of the difference between the denominators.

Proof. Let a/b be the given fraction, and let d be the denominator of the equivalent fraction. By the rule, the new fraction is obtained to be

$$\frac{a+\frac{a}{b}(d-b)}{d}$$

which reduces to the given fraction a/b.

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The Demonstration of Intersections

By W. V. Burg
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It is well known to every teacher of elementary descriptive geometry what serious difficulties the average student encounters in visualizing the more complicated geometrical figures that are produced by the intersecting of solids. Among the numerous devices that have been proposed to assist the beginner in this task only a few have become generally accepted. Besides drawings which picture the intersections perspectively, models of solid material (wood, metal, gypsum, etc.) or string models have been most widely employed. A distinct advantage of the latter over any non-transparent model lies in the fact that they show the line of intersection in its entirety and not only that portion of it which points towards the observer. Another advantage of these models is their flexible nature which makes it possible to vary the relative positions of the two solids. In this respect, however, string models are only partially satisfactory since their flexibility is necessarily limited by the given structural conditions. Moreover, the demonstration of intersections with the aid of such models is only feasible if the intersecting surfaces can be produced by straight lines. Considering these limitations it may be worth while to call attention to another method of representing intersections* which possesses several advantages over string models.

In this method only one of the intersecting surfaces is formed by a three-dimensional model while the other one is produced by passing light from a lantern through a slide that is entirely black with the exception of the geometrical figure to be projected. According to the shape of that figure and the manner of projection a wide variety of plane, cylindrical, conical, or other surfaces may be produced. If such a surface penetrates the model, which is made of some transparent material like white crinoline or tulle, the entire line of intersection will be clearly visible.

It is of importance that the method gives satisfactory results without requiring a complete darkening of the classroom. This enables the students to carry out the drawings while the intersection is being demonstrated. It is only necessary to protect the model from light other than that emitted by the lantern so that the curve of inter-

*Burg, German Patent 108302.

section will appear bright as contrasted with the dim outlines of the model. The surface which is produced by the beams from the lantern should be made visible by means of smoke or in a similar manner. It is, furthermore, of advantage to have that part of the curve which points away from the observer appear somewhat less bright than the front portion. The drawing of the model itself is preferably done before the slide is inserted and while the model is strongly illuminated from the lantern.

A few advantages of the method may be mentioned. For most of the common applications models and slides can be prepared by the students without difficulty and at little expense. With a few models and slides an infinite number of intersections can be produced since not only the manner of projection may be varied but also the relative positions of model and slide can be altered at will. An important feature of the method is that it permits to demonstrate the gradual penetration of one solid by another one, thus enabling the students to see the formation of the line of intersection. This is especially valuable with regard to the more complicated curves which can be developed by steps before the eyes of the class, starting with the simplest possible case.

Though the method just described is of particular value in the teaching of descriptive geometry, there are several other fields such as solid and analytic geometry in which it may be used with success. An interesting application is the demonstration of more difficult cinematic problems in courses on advanced mathematics.

Mathematical World News

Edited by
L. J. ADAMS

Dr. Thurman Stewart Peterson has been elected to an instructorship in the department of mathematics, University of Oregon. Dr. Peterson graduated with honors from California Institute of Technology with degree of B S. For three years he was a fellow at Ohio State University from which he received his M. S. and Ph.D. He was then for two years an instructor of the University of Michigan; following which he was two years a research worker at the Institute for Advanced Study of Princeton. More recently he has been head of mathematics and science at Shipley School, Bryn Mawr, Pennsylvania.

Professor H. A. Simmons sends the following news of the mathematics department at Northwestern University:

1. Miss Lois Griffiths has been promoted from the rank of assistant professor to that of associate professor.

2. Associate professors F. E. Wood and H. S. Wall are in residence again after leaves of absence last year. Professor Wall spent last year at the Institute for Advanced Study.

3. Professor A. W. Tucker of Princeton University, who taught at Northwestern last summer, recently lectured to the Northwestern Mathematics Club on *A topological Abelian group.*

4. To commemorate the lives and works of Josiah Willard Gibbs and George William Hill, Northwestern recently had a celebration of their one hundredth anniversaries. Professor R. E. Langer of the University of Wisconsin gave an address n honor of Josiah Willard Gibbs, and Mr. F. R. Moulton, formerly professor of astronomy at the University of Chicago and now permanent secretary of the A. A. A. S., gave an address in honor of George William Hill. After the lectures, the audience enjoyed open house with refreshments in the new quarters of the Northwestern Mathematics Department, Willard Hall.

5. Dr. W. T. Scott, formerly of the Rice Institute, is doing research work here on continued fractions, a subject in which Professor Wall has made numerous contributions.

6. During the second semester of this academic year, Professor D. R. Curtiss will be on leave of absence.

The following note concerns Princeton University: To meet the growing demand for men trained in mathematical statistics, the department, in cooperation with the department of economics, now gives three one-term courses: Introduction to Statistics, Elementary Mathematical Statistics, and Statistical Inference. "With this arrangement of courses and the opportunities for further study of the subject in the independent reading of junior and senior years," Dean Eisenhart says, "we have a program which should qualify a student very well in this field at the undergraduate evel."

The following news items from Mississippi State College come from Professor S. B. Murray: Mr. F. W. Gamblin goes to Mississippi State College this session as instructor in mathematics. Mr. Gamblin received his master's degree from the University of Mississippi, and before going to State College was on the faculty of the Holmes Junior College at Goodman, Miss. He replaces Mr. E. P. Coleman, who left during the summer to accept a place on the faculty of the University of Omaha, Nebraska. Doctors Arthur Ollivier and W. E. Cox have been promoted to the rank of assistant professor. Dr. Ollivier has recently published in Metron a paper on *Analysis of Variance of General Mortality Rates.*

During the annual meeting of the American Mathematical Society the section meetings are devoted to Analysis, Topology and Geometry, Calculus of Variations and Applied Mathematics and Algebra. At the separate session of the Mathematical Association of America held on December 31, the following addresses were presented:

1. *Limits to the characteristic roots of a matrix.* Professor E. T. Browne, University of North Carolina.
2. *Finite deformations of an elastic solid.* Professor F. D. Murnaghan, John Hopkins University.
3. *Differential geometry in the large.* Professor S. B. Myers, University of Michigan.

Professor G. Szego has been appointed executive head of the department of mathematics in Stanford University.

At the recent meeting of the American Mathematical Society in Los Angeles, Professor A. A. Shaw, University of Arizona, presented a discussion of the works of David the Invincible, one of the early Armenian philosopher-mathematicians. This meeting included several research papers, and was held at the University of California at Los

Angeles. After the meeting the members of the Society and their wives were guests at a tea in the home of Dr. E. R. Hedrick, vice-president of the university.

(Authorized republication of a reprint from *Bulletin of the American Mathematical Society*, September, 1938.)

THE INTERNATIONAL CONGRESS OF MATHEMATICIANS

CAMBRIDGE, MASSACHUSETTS, U. S. A.
SEPTEMBER 4-12, 1940

On invitation by the American Mathematical Society, an International Congress of Mathematicians will be held in Cambridge, Massachusetts, in 1940.

FORMER CONGRESSES. In connection with the World's Columbian Exposition in Chicago in 1893 there was an International Congress of Mathematicians. The first congress in Europe was held at Zurich in 1897, and the most recent one met in Oslo in 1936. In the interim, with a few omissions necessitated by the World War, there have been sessions about once every four years. These have all been in Europe except for that of 1924, which was held at the University of Toronto with the financial support of the province and the Dominion.

At recent congresses, the number of countries represented has been about 40, the number of members about 600, and the number of short papers submitted about 250.

TIME AND PLACE. The dates of the Congress have been fixed as September 4-12, 1940. The Society will omit its usual summer meeting in 1940. Harvard University and the Massachusetts Institute of Technology will be the local hosts of the Congress. Some neighboring institutions will join in the hospitality, but all institutions in the United States and Canada are invited to consider themselves as participants. Mathematicians so desiring will be housed in the Harvard University dormitories at modest rates, and meals will be served at cost in the University dining rooms. There will be accommodations for members of families, special provision being made for the care of children. Those who prefer to live in hotels can be comfortably provided for in Cambridge or Boston. It is expected that the Society will be able to furnish room and board without charge for a considerable number of foreign guests during the week that they are in Cambridge. In connection with the Harvard Tercentenary, the American Mathematical Society and the Mathematical Association of America met in Cambridge in September, 1936. There is no need of assuring any of the 800 people who were present then that excellent arrangements will be made for the Congress.

ORGANIZATION. Plans for the scientific features of the Congress are well under way. Following precedent, there will be a score of invited addresses, each of an hour's length, and sectional meetings for the presentation of short papers.

An innovation will be conferences, somewhat after the pattern of recent international gatherings in Moscow for Topology and in Zurich for Probability. Each conference will be devoted to some field in which a vigorous advance has recently been made or is currently in progress. The purpose will be the exchange of information and opinion by specialists in the field, and the dissemination of new and important results among the mathematical public at large. This will be accomplished by a coordinated program of formal lectures and informal open discussion. Among the conferences will be one on Algebra, with Professor A. A. Albert as chairman, one on the Theory of Measure, Probability, and Allied Topics, with Professor Norbert Wiener as chairman, one on

Mathematical Logic with Professor H. B. Curry as chairman, and one on Topology with Professor Solomon Lefschetz as chairman.

Six sections are tentatively planned for the presentation of papers: (I) Algebra and Theory of Numbers; (II) Analysis; (III) Geometry and Topology; (IV) Probability, Statistics, Actuarial Science, Economics; (V) Mathematical Physics and Applied Mathematics; (VI) Logic, Philosophy, History, Didactics. The International Commission on the Teaching of Mathematics proposes to have a session in connection with the Congress.

These short papers will be preferably in one of the official languages of the Congress (English, French, German, and Italian), and will not exceed ten minutes in length.

SOCIAL FEATURES, EXCURSIONS. There will be many interesting entertainment features provided—reception, garden party, symphony concert, and banquet. In addition there will be a number of automobile excursions, and it is hoped that American mathematicians will be able to assist in the entertainment by putting their automobiles at the disposal of the Entertainment Committee for trips to be made out of Cambridge.

Every effort will be made to facilitate the travel at reasonable cost of foreign participants while in the United States. Previous to the Congress, opportunity will be given them to see New York City under the guidance of some mathematician. It is expected that an international exposition will be in progress at that time in the city. Excursions to Washington, Niagara Falls, and other places will be arranged if there is demand.

MEMBERSHIP IN THE CONGRESS. Membership in the Congress will be open to all persons, whether they are able to be present in person or not. For regular members of the Congress, the fee is $10.00; these persons will receive the Proceedings of the Congress. Persons not participating in the scientific deliberations may become associate members, for whom the fee is $5.00; they will not present papers nor receive the Proceedings but will be entitled to many of the other privileges of membership.

FINANCIAL SUPPORT. Besides the support from Harvard University, Massachusetts Institute of Technology, and other institutions in the neighborhood, generous subventions have been subscribed for the Congress by the Carnegie Corporation, the Institute for Advanced Study, the National Research Council, and the Rockefeller Foundation.

INFORMATION. Detailed information will be sent in due course to all members of the American Mathematical Society. Others interested in receiving information may file their names in the Office of the Society, and such persons will receive from time to time information regarding the program and arrangements.

Communications should be addressed to the American Mathematical Society, 531 West 116th Street, New York City, U. S. A.

THE ORGANIZING COMMITTEE.

Problem Department

Edited by
ROBERT C. YATES and EMORY P. STARKE

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to ROBERT C. YATES, College Park, Md.

SOLUTIONS

Late Solutions: No. 15 by *H. T. R. Aude* and *C. A. Balof*.

No. 33. Proposed by *F. M. Kenny*, Malone, N. Y.

The trees in an orchard of rectangular form are arranged so that the rows are the same distance apart as the trees in the rows. Find the arrangement if half of the trees are in the four rows bordering the rectangle.

Solution by *D. L. MacKay*, Evander Childs High School.

If y rows each have x trees, then the number of trees in the four border rows is

$$2x+2(y-2)=xy/2,$$

or

$$x=4+8/(y-4).$$

This demands that y be less than 12 but greater than 4. Accordingly, there are but two sets of positive integer solutions: $x=12$, $y=5$ and $x=8$, $y=6$.

Also solved by *G. W. Wishard, W. B. Clarke, H. T. R. Aude, C. A. Balof, T. F. Mulcrone*, and *C. W. Trigg*.

The last two named solvers presented solutions under the interpretation that half of the trees were in four rows along each border. In this event,

$$2[4x+4(y-8)]=xy/2,$$

or

$$x=16(y-8)/(y-16).$$

The possible arrangements of the rectangular orchard here are four:

144x17, 80x18, 48x20, 32x24.

No. 112. Proposed by *Walter B. Clarke*, San Jose, California.

Show that the distance of the orthocenter from the longest side of a triangle equals the sum of the distances of incenter, verbicenter, and Nagel point from that side.

Solution by *C. A. Balof*, Lincoln College, Illinois.

In the triangle ABC let BC be the longest side; H' and H'' be the feet of the altitudes from A and C respectively; J' and J'' be the points half way around the triangle from A and B respectively; I', O', and V' be the feet of the perpendiculars to BC from the incenter I, the circumcenter O, and the verbicenter V, respectively; H be the orthocenter and N be the Nagel point.

Case I. Angle A less than 90°.

$AH' = c \sin B$, and $AH = AH'' \sec H'AB = b \cos A/\sin B$, whence $HH' = c \sin B - b \cos A/\sin B$.

$II' = (s-a)\tan \frac{1}{2}A$ where s is one-half the perimeter of ABC. In triangle ABJ', $AV/VJ' = c \sin ABV/BJ'(\sin VBC)$, and in triangle

$$ABC,\ AJ''/CJ'' = c \sin ABV/a \sin VBC,$$

whence $$AV/VJ' = (AJ''/CJ'')(a/BJ') = a/(s-a).$$

Then $AJ'/VJ' = s/(s-a)$. Triangles $AH'J'$ and $VV'J'$ are similar and $VV'/AH' = VJ'/AJ'$, whence $VV' = (s-a)c \sin B/s$.

The circumcenter, the incenter, and the Nagel point are collinear, the circumcenter being equidistant from the other two points. (See Altshiller-Court, *College Geometry*, p. 105). If N and O are on the same side of BC, $II'J'N$ is a trapezoid of which OO' is a median. Hence $NJ' = 2(OO') - II'$ or $NJ' = b \cos A/\sin B - (s-a)\tan \frac{1}{2}A$. Then $II' + VV' + NJ' = (s-a)c \sin B/s + b \cos A/\sin B = c \sin B - (b/\sin B)(\cos B + \cos C - 1)$ when reduced. Hence $II' + VV' + NJ' = HH'$ if $\cos B + \cos C - \cos A = 1$. But this is true only when NJ' is zero. Hence the proposition is not true if N is on the same side of BC as is O.

If N and O are on opposite sides of BC, $NJ = II' - 2(OO')$. Then $II' + VV' + NJ' = 2(s-a)\tan \frac{1}{2}A + (s-a)c \sin B/s - b \cos A/\sin B = c \sin B - b \cos A/\sin B = HH'$.

Case II. Angle A is 90°.

$HH'=c \sin B$, $II'=(s-a)$, $VV'=(s-a)c \sin B/s$, and $NJ'=II'$. Then $II'+VV'+NJ'=HH'$.

Case III. Angle A is greater than 90°.

HH', II', and VV' have the same values as in Case I.

$$NJ'=2(OO')+II',$$

whence $$NJ'=II'-b \cos A/\sin B.$$

Then as in Case I, $$II'+VV'+NJ'=HH'.$$

No. 237. Proposed by *E. P. Starke*, Rutgers University.

Show that no integer $x^2+y^2+z^2+1$ can be a multiple of 8.

Solution by *Fred Marer*, Los Angeles, California.

For $M=x^2+y^2+z^2+1$ to be even, x, y, and z must be (a) all odd or (b) one odd, two even. Under (a), x, y, and z are all of the form $4n\pm1$; then the division of M by 8 gives the remainder 4. Under (b), one letter is of the form $4n\pm1$ and the other two are of form $4n$ or $4n+2$; then the division of N by 8 gives the remainder 2 or 6. Hence no M exists for which the remainder is 0.

Also solved by *C. W. Trigg*.

No. 239. Proposed by *V. Thébault*, Le Mans, France.

What is the largest term in the expansion of $(3+4)^{34}$?

Solution by *C. W. Trigg*, Los Angeles City College.

In the expansion of $(p+q)^n$, the terms will successively increase up to $p^n \cdot {}_nC_r(q/p)^{r-1}$

where $$p^n \cdot {}_nC_{r+1}(q/p)^r \div p^n \cdot {}_nC_r(q/p)^{r-1}=({}_nC_{r+1}/{}_nC_r)(q/p)$$
$$=[(n-r)/(r+1)]\,(q/p)\leqq 1.$$

If this last expression is an equality, then there are two equal largest terms. We have $[(34-r)/(r+1)](4/3)\leqq 1$, and thus, in the expansion of $(3+4)^{34}$, the two largest terms are the twentieth and twenty-first, namely

$$3^{34}.\frac{34\cdot 33\cdots 16}{19!}\left(\frac{4}{3}\right)^{19}=3^{34}.\frac{34\cdot 33\cdots 15}{20!}\left(\frac{4}{3}\right)^{20}$$

$=1{,}391{,}975{,}640\cdot 4^{20}\cdot 3^{14}=732\ 03024\ 95327\ 11844\ 93484\ 44160.$

Also solved by *Fred Marer* and *Robert W. Clack*.

No. 240. Proposed by *G. W. Wishard*, Norwood, Ohio.

If A_1, B_1, and C_1 are 1, 5 and 7, respectively, A_1^2, B_1^2 and C_1^2 are in arithmetical progression with common difference $24=24\cdot 1$. If $A_2=C_1$, $B_2=B_1+8$, $C_2=C_1+10$, then A_2^2, B_2^2 and C_2^2 have a common difference $120=24(1^2+2^2)$. Similarly if $A_3=C_2$, $B_3=B_2+12$, $C_3=C_2+14$, then A_3^2, B_3^2 and C_3^2 have a common difference $336=24(1^2+2^2+3^2)$. Prove that the sequence extends to infinity and that A_n, B_n and C_n are relatively prime.

Solution by *C. W. Trigg*, Los Angeles City College.

The A's form the series $1, 7, 17, \cdots, (2n^2-1)$; the B's are $5, 13, 25, \cdots, (2n^2+2n+1)$; and the C's are $7, 17, 31, \cdots, (2n^2+4n+1)$. The general terms exhibited are easily found by Newton's method of finite differences or otherwise. Then $C_n^2-B_n^2=B_n^2-A_n^2=4n(2n+1)(n+1)$ $=24[n(n+1)(2n+1)/6]=24\sum_{k=1}^{n} k^2$. Hence the sequence extends to infinity.

If two numbers have a common factor, it is also a factor of their difference. $B_n-A_n=2(n+1)$ which has evidently no factor in common with $A_n=2(n+1)(n-1)+1$. Similarly, $C_n-B_n=2n$ and $C_n-A_n=2(2n+1)$; neither of these has a factor in common with $C_n=2n(n+2)+1=[(2n+1)(2n+3)-1]/2$, for any value of n. Therefore A_n, B_n and C_n are always relatively prime.

Also solved by the *Proposer*.

No. 242. Proposed by *E. C. Kennedy*, Texas College of Arts and Industries.

For how many positive integral values of B is $R=\sqrt{(B^2+N)}$ an integer, if $N=10395=1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11$? What are the two largest values of R?

Solution by *C. W. Trigg*, Los Angeles City College.

$N=R^2-B^2=(R+B)(R-B)=1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11=10395<102^2$. Hence there are as many integral values of B as there are factors of 10395 less than 102. Since $9\cdot 11<102<3\cdot 5\cdot 7$, there are ${}_6C_1+{}_5C_2$, or sixteen values of B. By setting the smaller factor equal to $R-B$, and the larger equal to $R+B$, sixteen pairs of simultaneous equations are secured. These when solved give the solutions for R and B. The largest values of R are thus 5198 and 1734.

Also solved by *Fred Marer* and the *Proposer*.

No. 246. Proposed by *Julius S. Miller*, Dillard University, New Orleans.

An old farmer had 30 children, 15 by his first wife, now deceased, and 15 by his second. This latter wife wished that her eldest son be heir to the property. Accordingly, she addressed her husband thus: "Let us settle this problem of who shall be your heir. We will arrange our 30 children in a circle, and remove every tenth child until there is left but one, who will succeed to your estate." The husband accepted the proposal, but as the process of elimination progressed he was bewildered to discover that the first 14 to be removed were his children by his first wife; and further, that the very next one to be eliminated would be her last child. In desperation he suggested that they count backwards beginning with this last child, in the hope that this child might be saved. The odds being apparently 15 to 1 in his wife's favor, she did not hesitate. Nevertheless, the farmer's son by his first wife became the heir. Required to know the arrangement of the children.

Solution by *T. F. Mulcrone*, Spring Hill College, Alabama.

The problem is an "oriental setting" of the famous Josephus Problem, a device by which Josephus (37-101, A. D.), the Jewish historian, is said to have saved his life when a band of forty Jews, disheartened at the capture of Jotaphat by the Romans under Vespasian (67 A. D.), decided that they should slay each other in turn rather than fall into the hands of the conqueror.

In the present problem the children must be arranged in the following order:

S S F S S S F F F F F S S F F S S S S F S F F F S F F S S F

where F represents a child by the first wife and S one by the second. The heir is No. 14, (F). The eldest son of the second wife (the intended heir) is No. 28, (S). The order may be recalled by the position of the vowels in the following mnemonic Latin verse: "Rex paphi cum gente bona dat signa serena", where *a* stands for 1, *e* for 2, *i* for 3, *o* for 4, and *u* for 5. Hence the order is $2(=e)$ children by the second wife, $1(=a)$ by the first wife, etc.

Such problems are solved empirically. There are, however, certain theorems which lessen the labor required. That is,

If with n individuals, subject to the elimination of every mth one, the last survivor occupied originally the p_1th place, then:

Theorem I. His original position is shifted forward along the circle m places for each addition of a single man to the original group; and,

Theorem II. With $(n+x)$ individuals the last survivor occupies the p_2-place, where the lowest value of x is taken that makes p_2 a positive number less than m in the equation,

$$p_2 = p_1 + mx - (n+x).$$

The usefulness of Theorem II is not evident until a solution is desired where n is very large. For by the use of Theorem I, n always increases by unity, but the larger n is, the farther apart will be the values of n found by substitution in II. Once we have found that value of n just less than the value we are interested in, then Theorem I applies at once. Here, however, n is increased, not by unity as before, but by the exact amount necessary to bring it up to the value with which we are concerned.*

Also solved by *Albert Farnell*, *C. W. Trigg*, and *G. W. Wishard*.

PROPOSALS

No. 267. Proposed by *Arnold Emch*, University of Illinois.

Theorem: Given an equilateral hyperbola, H, with vertices A and B. Join any point P on H to A and B. Prove that the bisecting line of the angle APB is parallel to one of the asymptotes.†

No. 268. Proposed by *H. T. R. Aude*, Colgate University.

For the equation $x^2+xy+y^2=k$, only the positive integer solutions with $x<y$ are to be considered. The problem is to find the least values of k such that the equation has, respectively, two, three, four, solutions. Also find these solutions.

No. 269. Proposed by *V. Thébault*, Le Mans, France.

Form a perfect square of four digits such that the sum of the two periods of two digits each will equal the number formed by reversing the digits of the root of the square.

No. 270. Proposed by *O. E. Lancaster*, University of Maryland.

Show that

$$n^{n-1}-(n-1)^n+(n-1)(n-2)^n/2!-(n-1)(n-2)(n-3)^n/3!+\cdots \pm(n-1)2^{n-1}\mp 1=(n-1)!$$

*An interesting account of the Josephus Problem, together with its solution and the mention of several similar problems, may be found in W. W. Ball, *Mathematical Recreations and Essays*, pp. 23-27.

†Does this appear anywhere in print?—ED.

No. 271. Proposed by *Robert C. Yates*, University of Maryland.

Given two distinct points A and B. Construct their midpoint by means of compasses alone.

No. 272. Proposed by *Walter B. Clarke*, San Jose, California.

Given the line segments of lengths d, e, f. Locate these as three chords concurrent at P in a circle of minimum size so that P divides each chord as the orthocenter would divide it if they were altitudes of a triangle. What are the sides of the corresponding triangle?

No. 273. Proposed by *M. S. Robertson*, Rutgers University.

Given the two real circles cutting orthogonally:

$$x^2+y^2+Dx+Ey+F=0, \quad x^2+y^2=F, \quad F>0,$$

and a straight line $Ax+By+C=0$, $C\neq 0$.

Show that

$$C^2(x^2+y^2+Dx+Ey+F)-F(Ax+By+C)^2=0$$

represents two distinct straight lines when, and only when, one or the other of the two following conditions holds:

(a) the given straight line is the radical axis of the two circles, or
(b) the given straight line is tangent to the second circle at a point of intersection of the two given circles.

Bibliography and Reviews

Edited by
P. K. SMITH and H. A. SIMMONS

A Text-Book of Convergence. By W. L. Ferrar. Oxford University Press, New York, 1938. vi+192 pages. $3.50.

The book is written primarily for university undergraduate students, and purposely excludes topics suitable for graduate work. No knowledge of mathematics beyond the elements of differential and integral calculus is presupposed, and Part I can be read without this.

In Part I the theory of convergence of series is developed to the extent usually found in the appropriate chapter of a typical algebra text, but with more detail.

Part II covers most of the remaining theory to be expected in an elementary text book, including absolute convergence, uniform convergence, multiplication of series, binomial, exponential, and logarithmic expansions, power series, and the integral tests. The order notation is introduced, and Tannery's Theorem is presented in a more general form than is usual. There are short chapters on double series and infinite products. Summation of divergent series is considered to the extent of defining and proving consistency for summation $(C, 1)$. Part II closes with a brief chapter on Fourier Series.

In the appendix the Dedekind theory is developed sufficiently to prove two propositions that were taken as assumptions in the first part of the book, viz.: (1) Any irrational number is the limit of a monotone sequence of rational numbers, and (2) a bounded sequence possesses a least upper bound.

Problems are numerous and have a considerable range of difficulty. Several features of the book are worthy of mention. A shorthand notation for such terms as "there exists", "such that", etc. is used consistently throughout the book. The word "divergent" is confined to series such that the sum of n terms becomes infinite as n increases without limit, and the term "non-convergent" is used to describe series which neither converge nor diverge. Certain of the hypotheses in some of the theorems are emphasized by bold type, while the form of other theorems are enlarged beyond their usual size by the inclusion of a statement that certain facts are not implied by the theorem. These last features seem unfortunate to the reviewer,

though perhaps they can be justified on pedagogical grounds. Altogether the book should be very useful.

Georgia School of Technology. GAINES B. LANG.

Practical Business Mathematics. By Justin H. Moore and Julio A. Mira. Longmans, Green and Co., New York, 1938. vii+238 pages. $1.00.

The book begins with the elementary processes of arithmetic, including short-cuts, factoring, and fractions. This is an excellent feature, since those who attempt to use mathematics are frequently hampered by the lack of ability to handle the simplest kind of arithmetic, such as addition of fractions.

There follows a brief chapter on "Use of Letters in Computations", which is a very good introduction to algebra—sufficient to enable the student to realize the meaning and usefulness of a formula.

The next topics treated are ratio and proportion, percentage, involution and evolution. In the discussion of involution and evolution a brief, but adequate, explanation is given of exponents, including zero, negative, and fractional exponents. This leads to an appropriate chapter on "Logarithms".

Two chapters entitled "Weights and Measures" and "Denominate Numbers and Mensuration" respectively, give tables of weights, measures and money, and explain how to reduce these quantities to higher or lower denominations. The latter chapter also gives rules for finding areas and volumes of elementary geometric figures.

With Chapter XI, "Simple Interest", begins the treatment of the mathematics of investment or finance. It is naturally followed by chapters on discount. After chapters on compound interest and present worth there is an excellent treatment of annuities with their applications.

The following six methods of depreciation are given: straight line method, fixed percentage of diminishing book value method, sinking fund method, annuity method, interest-on-investment method, sum of year's digits method.

One rather expects the chapter on "Permutations, Combinations, and Probability" to be followed by a treatment of life annuities and insurance, but is disappointed in this respect.

The Appendix includes tables of foreign and local weights and measures, and tables of square roots and cube roots, in addition to interest and annuity tables. The table of compound amounts is

given to eight decimal places for a wide variety of interest rates, and for all positive integral numbers of years up to 100. The present value and annuity tables are not nearly so complete and are given to only four decimal places. A good summary of symbols and formulas is also included.

On the whole the book is clearly written, and practical in its outlook. It is fairly complete but might well have included a discussion of stocks and bonds, insurance, and life annuities. These topics seem to the reviewer more important than some of the material included in the discussion of weights and measures. With these exceptions, the book contains adequate treatments of most of the topics that would ordinarily be desirable in an elementary course in business mathematics, for which it seems well adapted.

Not the least unusual features of the volume are its price and the fact that it is bound in paper. These features have certain advantages, but whether they outweigh the advantages of a more substantially bound book at a higher price is debatable. It will be interesting to see with what reception these features meet.

Washington University,
St. Louis, Missouri.

PAUL R. RIDER.

Elementary Mathematical Statistics. By W. D. Baten. John Wiley & Sons, New York, 1938. x+338 pages. $3.00.

A large number of statistics texts have come from the presses in recent years. Each has made a claim for its special place. Some claim to be cure-alls for most every field where statistics are being used today. Most of the introductory texts give so little of the theory until they hardly justify a mathematics department adoption. With the rapidly growing *statistical family*, it becomes necessary to determine wherein the additions to the field differ from the older texts.

The reviewer has been impressed favorably with Professor Baten's new book. According to the preface, the text was planned "primarily for students interested in statistics who have not studied differential and integral calculus." Where theory has required advanced mathematics, examples have been given to point out its plausibility. In spite of the omissions of some proofs, the author has managed to deal with the newer subjects with clarity, and has actually given perhaps enough of the mathematical background to enable the student to see why various rules have been laid down for him. Professor Baten has documented his book with numerous references. The text is a distinct

improvement over the guides for research workers which give only the rules.

"The book contains many problems for students to solve, because the author believes that one learns statistics to a great extent by solving problems designed to bring out the meaning of theory." The problems constitute a wide, well-balanced list with a sprinkling of thought-provoking questions interspersed among the routine drill type.

In the first chapter, charts and graphs are introduced. The author perhaps should have given a longer discussion of the use of logarithmic charts. In the second chapter, statistical averages are taken up. An excellent presentation of grouped and ungrouped data is given. With the introduction of measures of dispersion some idea of sampling is developed. When the student arrives at chapter eleven on sampling, he will not be in the "usual darkness."

Although the author is a mathematician and a research associate in an agricultural experiment station, he states his book "can be used by those interested in business and economics." He has included chapters on index numbers, trend analysis and analysis of time series.

In chapters on correlation, comparisons have been presented for distinguishing differences between the concepts of linear correlation, non-linear, and correlation based on a correlation ratio. Details of partial, multiple and tetrachoric correlation have been excellently presented.

A few of the other chapters treat the normal curve, permutations, combinations, and probability, Bernoulli distribution, analysis of variance, and standard errors of certain statistics. The tables included are (I) area under the normal curve, (II) ordinates of the normal curve, (III) G. W. Snedecor's table for the interpretation of analysis of variance, (IV) squares, square roots and reciprocals to 1000, (V) common logarithms, and (VI) natural logarithms.

For the pleasure of the student, answers have been given at the end of the book! This text appears to be quite teachable.

Agnes Scott College. HENRY A. ROBINSON.

Essentials of Engineering Mathematics. J. P. Ballantine. Prentice-Hall, Inc., New York. xi+502+76.

In preparing this book, the author had two main objectives in mind. The first one was novelty of presentation. The second was to unify the material by breaking down the more or less arbitrary com-

partments into which undergraduate mathematics through the calculus is usually divided.

The first objective was obtained in a remarkable degree. The author has found a surprising number of new ways in which to present familiar topics. In some instances novel forms of standard formulæ are introduced. This is particularly true in the treatment of analytical geometry. There may be some question as to whether too much novelty has been introduced; however, this reviewer does not feel competent to voice an opinion on the point.

The attempt to unify the subject matter has not been so successful. The result is principally a rearrangement of the classical compartments, although it is true that some of them have been subdivided and separated. Practically all of the analytical geometry is treated after the introduction of derivatives, and helpful use of this concept is frequently made.

The fifth chapter is rather unusual. It contains twenty-eight pages entitled "Approximation Calculus", which deals with areas, volumes, rates, zeros of a function, maxima and minima—each found by means of increment approximations. Such approximations are recognized approaches to the applications of derivatives, but they are seldom found so fully treated.

The style in which the book is written is decidedly unusual. Its flow of language is that of the classroom lecturer rather than that of the usual textbook. It is possible that such a style makes for easy reading on the part of the student, but it has resulted in a somewhat lengthy book.

As the author states in his preface, the title of the book is misleading in that "it hardly mentions an engineering topic and keeps strictly to mathematics". Hence it is a book that would be suitable in any college where mathematics through the calculus is taught.

For the benefit of the engineer the author has inserted at several places some excellent practical problems. For instance, the pertinent facts about a certain dam are given. From these facts it is required to find such diverse things as pressure on the face of the dam, the amount of impounded water, the work done in lowering the level of the water, and the time required for a given amount of rainfall to flow over the spillway.

Instructors who seek novelty in their classes and have class time to devote to teaching experiments would find it profitable to examine the book.

Georgia School of Technology. RALPH A. HEFNER.

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